

Math 417 – Sections 75 & 76 Solutions

2. To compute the integral:

$$\int_0^\infty \frac{x^a}{(x^2 + 1)^2} dx, \quad -1 < a < 3$$

where $x^a = \exp(a \ln x)$ consider the integral:

$$\int_C \frac{\exp(a \log z)}{(z^2 + 1)^2} dz$$

where the branch cut $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ is used for $\log z$ and C is the contour consisting of the contours:

$$\begin{aligned} \text{semicircle, } C_R : \quad & |z| = R, \quad 0 \leq \theta \leq \pi \\ \text{the line, } C_1 : \quad & z = re^{i\pi}, \quad \rho \leq r \leq R \\ \text{semicircle, } C_\rho : \quad & |z| = \rho, \quad 0 \leq \theta \leq \pi \\ \text{the line, } C_2 : \quad & z = re^{i0}, \quad \rho \leq r \leq R \end{aligned}$$

Note that we use an indented contour where we go around the point $z = 0$. Therefore,

$$\int_C f(z) dz = \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{\exp(a \log z)}{(z^2 + 1)^2} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{\exp(a \log z)}{(z^2 + 1)^2}$ are:

$$z_{1,2} = \pm i$$

Note that $z_1 = i$ lies inside C and that it is a pole of order 2. The residue at this point is:

$$\text{Res}_{z=z_1} f(z) = \phi'(z_1)$$

where the function $\phi(z)$ and its derivative is:

$$\begin{aligned} \phi(z) &= \frac{\exp(a \log z)}{(z + i)^2} \\ \phi'(z) &= \frac{\frac{a}{z}(z + i)^2 \exp(a \log z) - 2(z + i) \exp(a \log z)}{(z + i)^4} \\ &= \exp(a \log z) \cdot \frac{a(z + i) - 2z}{z(z + i)^3} \end{aligned}$$

Therefore, the residue is:

$$\text{Res}_{z=i} f(z) = \exp(a \log i) \cdot \frac{a(i + i) - 2i}{i(i + i)^3} = \exp\left(i \frac{a\pi}{2}\right) \cdot \frac{2i(a - 1)}{8} = \frac{i(a - 1)}{4} \exp\left(i \frac{a\pi}{2}\right)$$

Therefore,

$$\int_C \frac{\exp(a \log z)}{(z^2 + 1)^2} dz = 2\pi i \left[\frac{i(a - 1)}{4} \exp\left(i \frac{a\pi}{2}\right) \right] = \frac{\pi(1 - a)}{2} \exp\left(i \frac{a\pi}{2}\right)$$

II. Using the parametrization $z = re^{i\pi} = -r$, $\rho \leq r \leq R$ we have:

$$\int_{C_1} \frac{\exp(a \log z)}{(z^2 + 1)^2} dz = \int_R^\rho \frac{\exp(a(\ln r + i\pi))}{(r^2 + 1)^2} (-dr) = \int_\rho^R \frac{r^a \exp(i\pi)}{(r^2 + 1)^2} dr$$

III. Using the parametrization $z = re^{i0} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_2} \frac{\exp(a \log z)}{(z^2 + 1)^2} dz = \int_\rho^R \frac{r^a}{(r^2 + 1)^2} dr$$

IV. Now consider the integral over C_ρ . We determine an upper bound on the integrand on C_ρ as follows:

$$\left| \frac{\exp(a \log z)}{(z^2 + 1)^2} \right| \leq \frac{|z|^a}{||z|^2 - 1|^2} \leq \frac{\rho^a}{(1 - \rho^2)^2}$$

Using the ML -Bound, we have:

$$\left| \int_{C_\rho} \frac{\exp(a \log z)}{(z^2 + 1)^2} dz \right| \leq \frac{\rho^a}{(1 - \rho^2)^2} \cdot \pi\rho = \frac{\pi\rho^{a+1}}{(1 - \rho^2)^2}$$

Note that since $-1 < a < 3$, the quantity $1 + a$ is greater than 0. Therefore, when we take the limit as ρ goes to 0, the right hand side of the above inequality goes to 0.

V. Now consider the integral over C_R . We determine an upper bound on the integrand on C_R as follows:

$$\left| \frac{\exp(a \log z)}{(z^2 + 1)^2} \right| \leq \frac{|z|^a}{||z|^2 - 1|^2} \leq \frac{R^a}{(R^2 - 1)^2}$$

Using the ML -Bound, we have:

$$\left| \int_{C_R} \frac{\exp(a \log z)}{(z^2 + 1)^2} dz \right| \leq \frac{R^a}{(R^2 - 1)^2} \cdot \pi R = \frac{\pi R^{a+1}}{(R^2 - 1)^2}$$

Note that since $-1 < a < 3$, the quantity $1 + a$ is less than 4. Therefore, when we take the limit as R goes to ∞ , the right hand side of the above inequality goes to 0.

Putting it all together, as we take ρ to 0 and R to ∞ we have:

$$\begin{aligned} \int_C f(z) dz &= \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ \frac{\pi(1-a)}{2} \exp\left(i\frac{a\pi}{2}\right) &= 0 + 0 + \int_0^\infty \frac{r^a \exp(i\pi)}{(r^2 + 1)^2} dr + \int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr \\ (1 + \exp(i\pi)) \int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr &= \frac{\pi(1-a)}{2} \exp\left(i\frac{a\pi}{2}\right) \\ \int_0^\infty \frac{r^a}{(r^2 + 1)^2} dr &= \frac{\pi(1-a)}{2} \cdot \frac{\exp(i\frac{a\pi}{2})}{1 + \exp(i\pi)} \\ &= \frac{\pi(1-a)}{4} \cdot \frac{2}{\exp(i\frac{a\pi}{2}) + \exp(-i\frac{a\pi}{2})} \\ &= \frac{\pi(1-a)}{4} \cdot \frac{1}{\cos(\frac{a\pi}{2})} \end{aligned}$$

4. Consider the integral:

$$\int_C \frac{(\log z)^2}{z^2 + 1} dz$$

where the branch cut $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$ is used for $\log z$ and C is the contour consisting of the contours:

$$\begin{aligned} \text{semicircle, } C_R : \quad & |z| = R, \quad 0 \leq \theta \leq \pi \\ \text{the line, } C_1 : \quad & z = re^{i\pi}, \quad \rho \leq r \leq R \\ \text{semicircle, } C_\rho : \quad & |z| = \rho, \quad 0 \leq \theta \leq \pi \\ \text{the line, } C_2 : \quad & z = re^{i0}, \quad \rho \leq r \leq R \end{aligned}$$

Note that we use an indented contour where we go around the point $z = 0$. Therefore,

$$\int_C f(z) dz = \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{(\log z)^2}{z^2 + 1} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{(\log z)^2}{z^2 + 1}$ are:

$$z_{1,2} = \pm i$$

Note that $z_1 = i$ lies inside C and that it is a simple pole. The residue at this point is:

$$\text{Res}_{z=z_1} f(z) = \phi(z_1)$$

where the function $\phi(z)$ is:

$$\phi(z) = \frac{(\log z)^2}{z + i}$$

Therefore, the residue is:

$$\text{Res}_{z=i} f(z) = \frac{(\log i)^2}{i + i} = \frac{(i\frac{\pi}{2})^2}{2i} = -\frac{\pi^2}{8i}$$

Therefore,

$$\int_C \frac{(\log z)^2}{z^2 + 1} dz = 2\pi i \left(-\frac{\pi^2}{8i} \right) = -\frac{\pi^3}{4}$$

II. Using the parametrization $z = re^{i\pi} = -r$, $\rho \leq r \leq R$ we have:

$$\begin{aligned} \int_{C_1} \frac{(\log z)^2}{z^2 + 1} dz &= \int_R^\rho \frac{(\ln r + i\pi)^2}{r^2 + 1} (-dr) \\ &= \int_\rho^R \frac{(\ln r)^2 - \pi^2 + 2\pi i \ln r}{r^2 + 1} dr \\ &= \int_\rho^R \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_\rho^R \frac{dr}{r^2 + 1} + 2\pi i \int_\rho^R \frac{\ln r}{r^2 + 1} dr \end{aligned}$$

III. Using the parametrization $z = re^{i0} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_2} \frac{(\log z)^2}{z^2 + 1} dz = \int_\rho^R \frac{(\ln r)^2}{r^2 + 1} dr$$

IV. Now consider the integral over C_ρ . We determine an upper bound on the integrand on C_ρ as follows:

$$\left| \frac{(\log z)^2}{z^2 + 1} \right| \leq \frac{(|\ln \rho| + |i\theta|)^2}{||z|^2 - 1|} \leq \frac{(\ln \rho + \pi)^2}{1 - \rho^2}$$

Using the ML -Bound, we have:

$$\left| \int_{C_\rho} \frac{(\log z)^2}{z^2 + 1} dz \right| \leq \frac{(\ln \rho + \pi)^2}{1 - \rho^2} \cdot \pi \rho$$

When we take the limit as ρ goes to 0, the right hand side of the above inequality goes to 0.

V. Now consider the integral over C_R . We determine an upper bound on the integrand on C_R as follows:

$$\left| \frac{(\log z)^2}{z^2 + 1} \right| \leq \frac{(|\ln R| + |i\theta|)^2}{||z|^2 - 1|} \leq \frac{(\ln R + \pi)^2}{R^2 - 1}$$

Using the $M L$ -Bound, we have:

$$\left| \int_{C_R} \frac{(\log z)^2}{z^2 + 1} dz \right| \leq \frac{(\ln R + \pi)^2}{R^2 - 1} \cdot \pi R$$

When we take the limit as R goes to ∞ , the right hand side of the above inequality goes to 0.

Putting it all together, as we take ρ to 0 and R to ∞ we have:

$$\begin{aligned} \int_C f(z) dz &= \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ -\frac{\pi^3}{4} &= 0 + 0 + \int_0^\infty \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \int_0^\infty \frac{dr}{r^2 + 1} + 2\pi i \int_0^\infty \frac{\ln r}{r^2 + 1} dr + \int_0^\infty \frac{(\ln r)^2}{r^2 + 1} dr \\ -\frac{\pi^3}{4} &= 2 \int_0^\infty \frac{(\ln r)^2}{r^2 + 1} dr - \pi^2 \left(\frac{\pi}{2} \right) + 2\pi i \int_0^\infty \frac{\ln r}{r^2 + 1} dr \end{aligned} \quad (1)$$

Taking the real part of Equation (1) we have:

$$\begin{aligned} -\frac{\pi^3}{4} &= 2 \int_0^\infty \frac{(\ln r)^2}{r^2 + 1} dr - \frac{\pi^3}{2} \\ \Rightarrow \int_0^\infty \frac{(\ln r)^2}{r^2 + 1} dr &= \frac{\pi^3}{8} \end{aligned}$$

Taking the imaginary part of Equation (1) we have:

$$\begin{aligned} 0 &= 2\pi \int_0^\infty \frac{\ln r}{r^2 + 1} dr \\ \Rightarrow \int_0^\infty \frac{\ln r}{r^2 + 1} dr &= 0 \end{aligned}$$