

Math 417 – Section 77 Solutions

2. To compute the integral:

$$\int_0^\infty \frac{x^{1/3}}{(x+a)(x+b)} dx, \quad a > b > 0$$

consider the integral:

$$\int_C \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz$$

where the branch cut $0 < \arg z < 2\pi$ is used for $\log z$ and C is the contour consisting of the contours:

$$\begin{aligned} \text{the circle, } C_R : \quad & |z| = R \\ \text{the line, } C_1 : \quad & z = re^{i(2\pi)}, \quad \rho \leq r \leq R \\ \text{the circle, } C_\rho : \quad & |z| = \rho \\ \text{the line, } C_2 : \quad & z = re^{i0}, \quad \rho \leq r \leq R \end{aligned}$$

Note that we use an indented contour where we go around the point $z = 0$. Therefore,

$$\int_C f(z) dz = \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)}$ are:

$$z_{1,2} = -a, -b$$

Both points lie inside C and both are simple poles. The residues at these points are:

$$\begin{aligned} \operatorname{Res}_{z=z_1} f(z) &= \phi_1(z_1) \\ \operatorname{Res}_{z=z_2} f(z) &= \phi_2(z_2) \end{aligned}$$

where the functions $\phi_1(z)$ and $\phi_2(z)$ are:

$$\begin{aligned} \phi_1(z) &= \frac{\exp(\frac{1}{3} \log z)}{z+b} \\ \phi_2(z) &= \frac{\exp(\frac{1}{3} \log z)}{z+a} \end{aligned}$$

Therefore, the residues are:

$$\begin{aligned} \operatorname{Res}_{z=z_1} f(z) &= \frac{\exp(\frac{1}{3} \log(-a))}{-a+b} = \frac{\sqrt[3]{a} \exp(i\frac{\pi}{3})}{b-a} \\ \operatorname{Res}_{z=z_2} f(z) &= \frac{\exp(\frac{1}{3} \log(-b))}{-b+a} = \frac{\sqrt[3]{b} \exp(i\frac{\pi}{3})}{a-b} \end{aligned}$$

Therefore,

$$\int_C \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz = 2\pi i \left[\frac{\sqrt[3]{a} \exp(i\frac{\pi}{3})}{b-a} + \frac{\sqrt[3]{b} \exp(i\frac{\pi}{3})}{a-b} \right] = -2\pi i \exp\left(i\frac{\pi}{3}\right) \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b}$$

II. Using the parametrization $z = re^{i(2\pi)} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_1} \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz = \int_R^\rho \frac{\exp(\frac{1}{3}(\ln r + 2\pi i))}{(r+a)(r+b)} dr = - \int_\rho^R \frac{\sqrt[3]{r} \exp(i\frac{2\pi}{3})}{(r+a)(r+b)} dr$$

III. Using the parametrization $z = re^{i0} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_2} \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr$$

IV. Now consider the integral over C_ρ . We determine an upper bound on the integrand on C_ρ as follows:

$$\left| \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} \right| \leq \frac{|z|^{1/3}}{(|z|-|a|)(|z|-|b|)} \leq \frac{\rho^{1/3}}{(a-\rho)(b-\rho)}$$

Using the ML -Bound, we have:

$$\left| \int_{C_\rho} \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz \right| \leq \frac{\rho^{1/3}}{(a-\rho)(b-\rho)} \cdot \pi\rho = \frac{\pi\rho^{4/3}}{(a-\rho)(b-\rho)}$$

When we take the limit as ρ goes to 0, the right hand side of the above inequality goes to 0.

V. Now consider the integral over C_R . We determine an upper bound on the integrand on C_R as follows:

$$\left| \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} \right| \leq \frac{|z|^{1/3}}{(|z|-|a|)(|z|-|b|)} \leq \frac{R^{1/3}}{(R-a)(R-b)}$$

Using the ML -Bound, we have:

$$\left| \int_{C_R} \frac{\exp(\frac{1}{3} \log z)}{(z+a)(z+b)} dz \right| \leq \frac{R^{1/3}}{(R-a)(R-b)} \cdot \pi R = \frac{\pi R^{4/3}}{(R-a)(R-b)}$$

When we take the limit as R goes to ∞ , the right hand side of the above inequality goes to 0.

Putting it all together, as we take ρ to 0 and R to ∞ we have:

$$\begin{aligned} \int_C f(z) dz &= \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ -2\pi i \exp\left(i\frac{\pi}{3}\right) \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} &= 0 + 0 - \int_0^\infty \frac{\sqrt[3]{r} \exp(i\frac{2\pi}{3})}{(r+a)(r+b)} dr + \int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr \\ -2\pi i \exp\left(i\frac{\pi}{3}\right) \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} &= \left[1 - \exp\left(i\frac{2\pi}{3}\right)\right] \int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr \\ \int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)} dr &= -2\pi i \cdot \frac{\exp\left(i\frac{\pi}{3}\right)}{1 - \exp\left(i\frac{2\pi}{3}\right)} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \\ &= \pi \cdot \frac{2i}{\exp(i\frac{\pi}{3}) - \exp(-i\frac{\pi}{3})} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \\ &= \pi \cdot \frac{1}{\sin \frac{\pi}{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \\ &= \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a} - \sqrt[3]{b}}{a-b} \end{aligned}$$

6. To compute the integral:

$$\int_0^\infty \frac{x^{-1/2}}{x^2+1} dx$$

consider the integral:

$$\int_C \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz$$

where the branch cut $0 < \arg z < 2\pi$ is used for $\log z$ and C is the contour consisting of the contours:

$$\begin{aligned} \text{the circle, } C_R : & |z| = R \\ \text{the line, } C_1 : & z = re^{i(2\pi)}, \quad \rho \leq r \leq R \\ \text{the circle, } C_\rho : & |z| = \rho \\ \text{the line, } C_2 : & z = re^{i0}, \quad \rho \leq r \leq R \end{aligned}$$

Note that we use an indented contour where we go around the point $z = 0$. Therefore,

$$\int_C f(z) dz = \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1}$ are:

$$z_{1,2} = \pm i$$

Both points lie inside C and both are simple poles. The residues at these points are:

$$\begin{aligned} \text{Res}_{z=z_1} f(z) &= \phi_1(z_1) \\ \text{Res}_{z=z_2} f(z) &= \phi_2(z_2) \end{aligned}$$

where the functions $\phi_1(z)$ and $\phi_2(z)$ are:

$$\begin{aligned} \phi_1(z) &= \frac{\exp(-\frac{1}{2} \log z)}{z + i} \\ \phi_2(z) &= \frac{\exp(-\frac{1}{2} \log z)}{z - i} \end{aligned}$$

Therefore, the residues are:

$$\begin{aligned} \text{Res}_{z=z_1} f(z) &= \frac{\exp(-\frac{1}{2} \log(i))}{i + i} = \frac{\exp(-i\frac{\pi}{4})}{2i} \\ \text{Res}_{z=z_2} f(z) &= \frac{\exp(-\frac{1}{2} \log(-i))}{-i - i} = \frac{\exp(-i\frac{3\pi}{4})}{-2i} \end{aligned}$$

Therefore,

$$\int_C \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz = 2\pi i \left[\frac{\exp(-i\frac{\pi}{4})}{2i} + \frac{\exp(-i\frac{3\pi}{4})}{-2i} \right] = \pi\sqrt{2}$$

II. Using the parametrization $z = re^{i(2\pi)} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_1} \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz = \int_R^\rho \frac{\exp(-\frac{1}{2}(\ln r + 2\pi i))}{r^2 + 1} dr = \int_\rho^R \frac{dr}{\sqrt{r}(r^2 + 1)}$$

III. Using the parametrization $z = re^{i0} = r$, $\rho \leq r \leq R$ we have:

$$\int_{C_2} \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz = \int_\rho^R \frac{dr}{r^2 + 1}$$

IV. Now consider the integral over C_ρ . We determine an upper bound on the integrand on C_ρ as follows:

$$\left| \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} \right| \leq \frac{|z|^{-1/2}}{(|z|^2 - |1|)} \leq \frac{1}{\sqrt{\rho}(1 - \rho^2)}$$

Using the $M L$ -Bound, we have:

$$\left| \int_{C_\rho} \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz \right| \leq \frac{1}{\sqrt{\rho}(1 - \rho^2)} \cdot \pi \rho = \frac{\pi \sqrt{\rho}}{1 - \rho^2}$$

When we take the limit as ρ goes to 0, the right hand side of the above inequality goes to 0.

V. Now consider the integral over C_R . We determine an upper bound on the integrand on C_R as follows:

$$\left| \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} \right| \leq \frac{|z|^{-1/2}}{(|z|^2 - |1|)} \leq \frac{1}{\sqrt{R}(R^2 - 1)}$$

Using the $M L$ -Bound, we have:

$$\left| \int_{C_R} \frac{\exp(-\frac{1}{2} \log z)}{z^2 + 1} dz \right| \leq \frac{1}{\sqrt{R}(R^2 - 1)} \cdot \pi R = \frac{\pi \sqrt{R}}{R^2 - 1}$$

When we take the limit as R goes to ∞ , the right hand side of the above inequality goes to 0.

Putting it all together, as we take ρ to 0 and R to ∞ we have:

$$\begin{aligned} \int_C f(z) dz &= \int_{C_R} f(z) dz + \int_{C_\rho} f(z) dz + \int_{C_1} f(z) dz + \int_{C_2} f(z) dz \\ \pi \sqrt{2} &= 0 + 0 + \int_0^\infty \frac{dr}{\sqrt{r}(r^2 + 1)} dr + \int_0^\infty \frac{dr}{\sqrt{r}(r^2 + 1)} dr \\ \int_0^\infty \frac{dr}{\sqrt{r}(r^2 + 1)} &= \frac{\pi \sqrt{2}}{2} \end{aligned}$$