

Math 417 – Section 78 Solutions

1. To evaluate the integral:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta} = \frac{1}{5} \int_0^{2\pi} \frac{d\theta}{1 + \frac{4}{5} \sin \theta}$$

we will use the result from the example on p. 279:

$$\int_0^{2\pi} \frac{d\theta}{1 + a \sin \theta} = \frac{2\pi}{\sqrt{1 - a^2}} \quad (-1 < a < 1)$$

with $a = \frac{4}{5}$. Therefore,

$$\frac{1}{5} \int_0^{2\pi} \frac{d\theta}{1 + \frac{4}{5} \sin \theta} = \frac{1}{5} \cdot \frac{2\pi}{\sqrt{1 - (\frac{4}{5})^2}} = \frac{2\pi}{3}$$

2. To evaluate the integral:

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}$$

we will integrate over the circle $|z| = 1$ oriented counterclockwise and use the substitutions:

$$\sin \theta = \frac{z - \bar{z}}{2i}, \quad d\theta = \frac{dz}{iz}$$

Therefore,

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} &= \int_C \frac{1}{1 + (\frac{z - \bar{z}}{2i})^2} \cdot \frac{dz}{iz} \\ &= 4i \int_C \frac{z}{z^4 - 6z^2 + 1} dz \end{aligned}$$

The singular points of the integrand are the values of z that satisfy:

$$z^4 - 6z^2 + 1 = 0$$

The solutions are:

$$z_{1,2} = \pm \sqrt{3 - 2\sqrt{2}}, \quad z_{3,4} = \pm \sqrt{3 + 2\sqrt{2}}$$

obtained using the quadratic equation. Note that $z_{1,2}$ lie inside C . The residues at these points are:

$$\begin{aligned} \operatorname{Res}_{z=z_1} f(z) &= \frac{z_1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} \\ \operatorname{Res}_{z=z_2} f(z) &= \frac{z_2}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} \end{aligned}$$

After some algebra, it can be shown that the sum of the residues is:

$$\begin{aligned} \operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z) &= \frac{1}{z_1^2 - z_3^2} \\ &= \frac{1}{3 - 2\sqrt{2} - 3 - 2\sqrt{2}} \\ &= -\frac{1}{4\sqrt{2}} \end{aligned}$$

The value of the integral is then:

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} &= 4i \int_C \frac{z}{z^4 - 6z^2 + 1} dz \\ &= 4i(2\pi i) \left(-\frac{1}{4\sqrt{2}} \right) \\ &= \pi\sqrt{2}\end{aligned}$$