Math 180, Exam 1, Fall 2013
Problem 1 Solution

1. Calculate each limit below.

(a) \( \lim_{x \to 7} \left( \frac{14}{x^2 - 7x} - \frac{2}{x - 7} \right) \)

(b) \( \lim_{x \to \infty} \frac{19x^4 + 2x - 1}{3x^4 + 16x^2 + 100} \)

Solution:

(a) The least common denominator of the function is \( x^2 - 7x \). Thus, the function can be written as follows:

\[
f(x) = \frac{14}{x^2 - 7x} - \frac{2}{x - 7} = \frac{14}{x(x - 7)} - \frac{2x}{x(x - 7)} = \frac{14 - 2x}{x(x - 5)} = \frac{-2(x - 7)}{x(x - 7)} = -\frac{2}{x}
\]

provided that \( x \neq 7 \). Therefore, the limit of \( f(x) \) as \( x \to 7 \) is

\[
\lim_{x \to 7} \left( \frac{14}{x^2 - 7x} - \frac{2}{x - 7} \right) = \lim_{x \to 7} \left( -\frac{2}{x} \right) = -\frac{2}{7}
\]

(b) The function is rational and the degrees of the numerator and denominator are the same. Therefore, the limit of \( f \) as \( x \to \infty \) is the ratio of the leading coefficients.

\[
\lim_{x \to \infty} \frac{19x^4 + 2x - 1}{3x^4 + 16x^2 + 100} = \frac{19}{3}
\]
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Problem 2 Solution

2. If \( f(x) = \sqrt{3x+1} \), calculate

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]

**Solution:** It is easiest to calculate the limit by recognizing that, by definition,

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x).
\]

Given that \( f(x) = \sqrt{2x + 1} \), we can use the Chain Rule:

\[
f'(x) = \frac{d}{dx} \sqrt{3x + 1} = \frac{1}{2\sqrt{3x + 1}} \cdot \frac{d}{dx}(3x + 1) = \frac{1}{2\sqrt{3x + 1}} \cdot 3
\]

The other method which almost every student used was to set up and evaluate the limit directly. Here’s the calculation:

\[
\begin{align*}
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \lim_{h \to 0} \frac{\sqrt{3(x + h) + 1} - \sqrt{3x + 1}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{3x + 3h + 1} - \sqrt{3x + 1}}{h} \cdot \frac{\sqrt{3x + 3h + 1} + \sqrt{3x + 1}}{\sqrt{3x + 3h + 1} + \sqrt{3x + 1}} \\
&= \lim_{h \to 0} \frac{(3x + 3h + 1) - (3x + 1)}{h(\sqrt{3x + 3h + 1} + \sqrt{3x + 1})} \\
&= \lim_{h \to 0} \frac{3h}{h(\sqrt{3x + 3h + 1} + \sqrt{3x + 1})} \\
&= \lim_{h \to 0} \frac{3}{\sqrt{3x + 3h + 1} + \sqrt{3x + 1}} \\
&= \frac{3}{\sqrt{3x + 3(0) + 1} + \sqrt{3x + 1}} \\
&= \frac{3}{2\sqrt{3x + 1}}
\end{align*}
\]
3. Let \( y = e^{2x} \cos(x) \). Find \( y'' \). You do not need to simplify your answers!

(b) Rewrite \( \tan(x) \) in terms of \( \sin(x) \) and \( \cos(x) \) and use the quotient rule to show that \( \frac{d}{dx} \tan(x) = \sec^2(x) \).

(c) Find \( \frac{d}{d\theta} \cot(\sin(\theta) + 3\theta^4) \).

Solution:

(a) Using the Product and Chain Rules, the first derivative is

\[
y' = e^{2x} \cdot \frac{d}{dx} \cos(x) + \cos(x) \cdot \frac{d}{dx} e^{2x}
\]

\[
y' = e^{2x} \cdot (-\sin(x)) + \cos(x) \cdot (2e^{2x})
\]

\[
y' = -e^{2x} \cdot \sin(x) + 2e^{2x} \cdot \cos(x)
\]

\[
y' = e^{2x} \cdot (-\sin(x) + 2 \cos(x))
\]

Another application of the Product and Chain Rules yields the second derivative:

\[
y'' = e^{2x} \cdot \frac{d}{dx} (-\sin(x) + 2 \cos(x)) + (-\sin(x) + 2 \cos(x)) \cdot \frac{d}{dx} e^{2x}
\]

\[
y'' = e^{2x} \cdot (-\cos(x) - 2 \sin(x)) + (-\sin(x) + 2 \cos(x)) \cdot (2e^{2x})
\]

\[
y'' = -e^{2x} \cdot \cos(x) - 2e^{2x} \cdot \sin(x) - 2e^{2x} \cdot \sin(x) + 4e^{2x} \cdot \cos(x)
\]

\[
y'' = -2e^{-x} \cos(x)
\]

(b) By definition,

\[
\tan(x) = \frac{\sin(x)}{\cos(x)}.
\]

Using the Quotient Rule yields

\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right)
\]

\[
\frac{d}{dx} \tan(x) = \frac{\cos(x) \cdot \frac{d}{dx} \sin(x) - \sin(x) \cdot \frac{d}{dx} \cos(x)}{\cos^2(x)}
\]

\[
\frac{d}{dx} \tan(x) = \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)}
\]

\[
\frac{d}{dx} \tan(x) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
\]

\[
\frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)}
\]

\[
\frac{d}{dx} \tan(x) = \sec^2(x)
\]
(c) Using the Chain Rule we have:

\[
\frac{d}{d\theta} \cot(\sin \theta + 3\theta^4) = -\csc^2(\sin \theta + 3\theta^4) \cdot \frac{d}{d\theta}(\sin \theta + 3\theta^4)
\]

\[
\frac{d}{d\theta} \cot(\sin \theta + 3\theta^4) = -\csc^2(\sin \theta + 3\theta^4) \cdot (\cos \theta + 12\theta^3)
\]
4. Let \( f \) be defined by
\[
 f(x) = \begin{cases} 
 x^4 + (1 + A)e^x, & \text{if } x < 0 \\
 -B, & \text{if } x = 0 \\
 \sin(x), & \text{if } x > 0 
\end{cases}
\]
where \( A \) and \( B \) are constants. Find values for \( A \) and \( B \) such that \( f \) is continuous on \(( -\infty, \infty)\) or state that no such constants exist. Justify your answer.

**Solution:** First, the function \( x^4 + (1 + A)e^x \) is continuous on \( x < 0 \) for any value \( A \). Second, the function \( \sin(x) \) is continuous on \( x > 0 \).

We must ensure that \( f \) is continuous at \( x = 0 \). That is, we must select \( A \) and \( B \) so that
\[
 \lim_{x \to 0} f(x) = f(0)
\]
The limit exists when the one-sided limits are the same.
\[
 \begin{align*}
 \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \sin(x) = \sin(0) = 0 \\
 \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x^4 + (1 + A)e^x) = 0^4 + (1 + A)e^0 = 1 + A
\end{align*}
\]
These limits are the same when \( A = -1 \) and in both cases, the limit is 0. Since \( f(0) = -B \) we must then have \( B = 0 \) for continuity at \( x = 0 \).
5. Assume the tangent line to the graph of \( f \) at \( x = 1 \) is given by
\[
    y = 4x + 2.
\]

(a) Find \( f(1) \).
(b) Find \( f'(1) \).
(c) Now assume that a function \( g \) is defined by \( g(x) = f(x^3) \). Find \( g(1) \) and \( g'(1) \).

Solution:

(a) When \( x = 1 \), the \( y \)-coordinate of the point on the tangent line is
\[
    y = 4(1) + 2 = 6
\]
Since the line is tangent to the graph of \( f \) at \( x = 1 \), we know that the point \((1, 6)\) is common to both graphs. Thus, \( f(1) = 6 \).

(b) The quantity \( f'(1) \) is the slope of the tangent line. Thus, \( f'(1) = 4 \).

(c) We know that \( g(1) = f(1^3) = f(1) = 6 \) (see part (a)).

To obtain \( g'(1) \) we begin by writing an expression for \( g'(x) \) using the Chain Rule.
\[
    g'(x) = \frac{d}{dx} f(x^3) = f'(x^3) \cdot \frac{d}{dx} x^3 = f'(x^3) \cdot 3x^2
\]
When \( x = 1 \) we have
\[
    g'(1) = f'(1^3) \cdot 3(1)^2 = f'(1) \cdot 3 = 4 \cdot 3 = 12
\]