Math 180, Exam 1, Practice Fall 2009
Problem 1 Solution

1. Evaluate the following limits, or show they do not exist.

(a) $\lim_{x\to\pi} 2\cos x$

(b) $\lim_{x\to 2} \dfrac{x^2-4}{x+2}$

(c) $\lim_{x\to 9} \dfrac{2-\sqrt{x-5}}{x-9}$

Solution:

(a) The function $f(x) = 2\cos x$ is continuous at $x = \pi$. In fact, $f(x)$ is continuous at all x in the interval $(-\infty, \infty)$. Therefore, we can evaluate the limit using substitution.

$$\lim_{x\to\pi} 2\cos x = 2\cos\pi = \boxed{-2}$$

(b) The function $f(x) = \dfrac{x^2-4}{x+2}$ is continuous at $x = 2$. In fact, $f(x)$ is continuous at all $x \neq -2$. Therefore, we can evaluate the limit using substitution.

$$\lim_{x\to 2} \frac{x^2-4}{x+2} = \frac{2^2-4}{2+2} = \boxed{0}$$

(c) When substituting $x = 9$ into the function $f(x) = \dfrac{2-\sqrt{x-5}}{x-9}$ we find that

$$\frac{2-\sqrt{x-5}}{x-9} = \frac{2-\sqrt{9-5}}{9-9} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by multiplying $f(x)$ by the "conjugate" of the numerator divided by itself.

$$\begin{aligned}
\lim_{x\to 9} \frac{2-\sqrt{x-5}}{x-9} &= \lim_{x\to 9} \frac{2-\sqrt{x-5}}{x-9} \cdot \frac{2+\sqrt{x-5}}{2+\sqrt{x-5}} \\
&= \lim_{x\to 9} \frac{4-(x-5)}{(x-9)(2+\sqrt{x-5})} \\
&= \lim_{x\to 9} \frac{-(x-9)}{(x-9)(2+\sqrt{x-5})} \\
&= \lim_{x\to 9} \frac{-1}{2+\sqrt{x-5}} \\
&= \frac{-1}{2+\sqrt{9-5}} \\
&= \boxed{-\frac{1}{4}}
\end{aligned}$$

We evaluated the limit above by substituting $x = 9$ into the function $\frac{-1}{2+\sqrt{x-5}}$. This is possible because the function is continuous at $x = 9$.

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Problem 2 Solution

2. Determine the location and type (removable, jump, infinite, or other) of all discontinuities of the function $\dfrac{x^2-3x+2}{x^2-1}$.

Solution: We start by factoring the numerator and denominator.

$$\frac{x^2-3x+2}{x^2-1} = \frac{(x-2)(x-1)}{(x+1)(x-1)}$$

As $x \to -1^+$, we find that:

$$\begin{aligned}\lim_{x\to-1^+}\frac{x^2-3x+2}{x^2-1} &= \lim_{x\to-1^+}\frac{(x-2)(x-1)}{(x+1)(x-1)}\\ &= \lim_{x\to-1^+}\frac{x-2}{x+1}\\ &= -\infty\end{aligned}$$

Therefore, $x=-1$ is an infinite discontinuity.

The limit at $x=1$ is:

$$\begin{aligned}\lim_{x\to1}\frac{x^2-3x+2}{x^2-1} &= \lim_{x\to1}\frac{(x-2)(x-1)}{(x+1)(x-1)}\\ &= \lim_{x\to1}\frac{x-2}{x+1}\\ &= \frac{1-2}{1+1}\\ &= -\frac{1}{2}\end{aligned}$$

However, $f(1)$ does not exist. Using our textbook's definitions, $x=1$ cannot be categorized as a removable, jump, or infinite discontinuity. Therefore, $x=1$ falls under the "other" category.

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Problem 3 Solution

3. Find the equation of the tangent line to $y = x^3 - 2x^2 + 2$ at $x = 1$.

Solution: The derivative y' is found using the Power Rule.

$$y' = (x^3 - 2x^2 + 2)' = 3x^2 - 4x$$

At $x = 1$ the values of y and y' are:

$$\begin{aligned} y(1) &= 1^3 - 2(1)^2 + 2 = 1 \\ y'(1) &= 3(1)^2 - 4(1) = -1 \end{aligned}$$

We now know that the point $(1, 1)$ is on the tangent line and that the slope of the tangent line is -1. Therefore, an equation for the tangent line in point-slope form is:

$$\boxed{y - 1 = -(x - 1)}$$

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Problem 4 Solution

4. Determine the value of c so that the function

$$f(x) = \begin{cases} 3cx + 1 & \text{if } x < 1 \\ 5x^2 + c & \text{if } x \geq 1 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution: The functions $3cx + 1$ and $5x^2 + c$ are continuous for all x. In order for $f(x)$ to be continuous on $\mathbb{R}$, we must select c so that $f(x)$ is continuous at $x = 1$. To do this, we must compute the one-sided limits at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3cx + 1) = 3c(1) + 1 = 3c + 1$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (5x^2 + c) = 5(1)^2 + c = 5 + c$$

In order to have continuity at $x = 1$, the one-sided limits must be equal there. Thus, we need:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1+} f(x)$$
$$3c + 1 = 5 + c$$
$$2c = 4$$
$$\boxed{c = 2}$$

For this value of c we have $\lim_{x\to 1} f(x) = 7$. Furthermore, we have $f(1) = 5(1)^2 + 2 = 7$. Thus, since $\lim_{x\to 1} f(x) = f(1)$ we know that $f(x)$ is continuous at $x = 1$.

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Problem 5 Solution

5. Use the Intermediate Value Theorem in order to show that the equation

$$x^5 - x + 1 = 0$$

has at least one real solution.

Solution: Let $f(x) = x^5 - x + 1$. First we recognize that $f(x)$ is continuous everywhere because it is a polynomial. Next, we must find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Let's choose $a = -2$ and $b = -1$.

$$\begin{aligned} f(-2) &= (-2)^5 - (-2) + 1 = -29 \\ f(-1) &= (-1)^5 - (-1) + 1 = 1 \end{aligned}$$

Since $f(-2) < 0$ and $f(-1) > 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $[-2, -1]$.

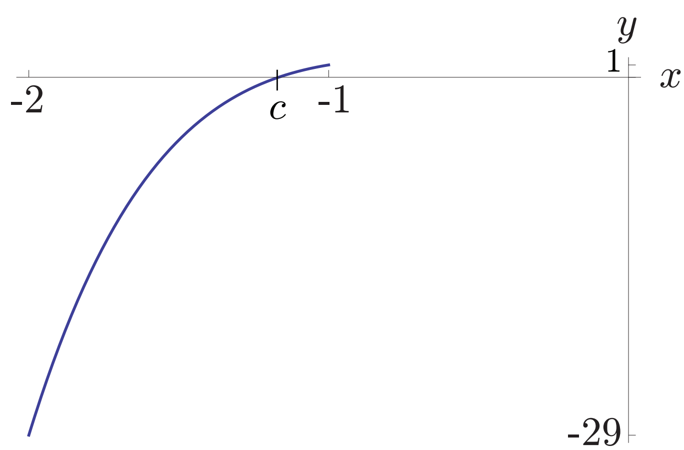

Figure 1: Graph of $f(x) = x^5 - x + 1$ on the interval $[-2, -1]$.

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Problem 6 Solution

6. Use the $\delta - \varepsilon$ definition of the limit to prove that $\lim_{x\to 3} 3x - 1 = 8$.

Solution: To show that $\lim_{x\to 3} 3x - 1 = 8$ we must find a $\delta > 0$ such that $|(3x-1)-8| < \varepsilon$ whenever $|x-3| < \delta$ for a given $\varepsilon > 0$.

Let's work with the inequality $|(3x-1)-8| < \varepsilon$.

$$\begin{aligned}
|(3x-1)-8| &< \varepsilon \\
|3x-9| &< \varepsilon \\
3|x-3| &< \varepsilon \\
|x-3| &< \frac{\varepsilon}{3}
\end{aligned}$$

Therefore, we choose $\delta = \dfrac{\varepsilon}{3}$.

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Problem 7 Solution

7. Let $f(x) = \dfrac{1}{x+1}$.

(a) Write the derivative, $f'(3)$, as the limit of the difference quotient.

(b) Evaluate this limit to find $f'(3)$.

Solution:

(a) There are two possible difference quotients we can use to evaluate $f'(3)$. One is:

$$f'(3) = \lim_{h\to 0} \frac{f(h+3) - f(3)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(h+3)+1} - \dfrac{1}{3+1}}{h}.$$

The other is:

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x-3} = \lim_{x\to 3} \frac{\dfrac{1}{x+1} - \dfrac{1}{3+1}}{x-3}$$

(b) Evaluating the first limit above we have:

$$\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{\dfrac{1}{(h+3)+1} - \dfrac{1}{3+1}}{h} \cdot \frac{4(h+4)}{4(h+4)} \\
&= \lim_{h\to 0} \frac{4-(h+4)}{4h(h+4)} \\
&= \lim_{h\to 0} \frac{-h}{4h(h+4)} \\
&= \lim_{h\to 0} \frac{-1}{4(h+4)} \\
&= \frac{-1}{4(0+4)} \\
&= \boxed{-\frac{1}{16}}
\end{aligned}$$

Evaluating the second limit we have:

$$\begin{aligned}
f'(3) &= \lim_{x\to 3} \frac{\dfrac{1}{x+1} - \dfrac{1}{3+1}}{x-3} \cdot \frac{4(x+1)}{4(x+1)} \\
&= \lim_{x\to 3} \frac{4-(x+1)}{4(x+1)(x-3)} \\
&= \lim_{x\to 3} \frac{-(x-3)}{4(x+1)(x-3)} \\
&= \lim_{x\to 3} \frac{-1}{4(x+1)} \\
&= \frac{-1}{4(3+1)} \\
&= \boxed{-\frac{1}{16}}
\end{aligned}$$

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Problem 8 Solution

8. Find the derivatives of the following functions using the basic rules. Leave your answers in an unsimplified form so that your method is obvious.

(a) $f(x) = x^3 + x^{-1} - x^{1/3}$

(b) $g(x) = x^3 e^x$

(c) $h(x) = \dfrac{3x}{1+x^2}$

Solution:

(a) Use the Power Rule.

$$f'(x) = \boxed{3x^2 - x^{-2} - \frac{1}{3}x^{-2/3}}$$

(b) Use the Product Rule.

$$\begin{aligned} g'(x) &= x^3(e^x)' + (x^3)'e^x \\ &= \boxed{x^3e^x + 3x^2e^x} \end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned} h'(x) &= \frac{(1+x^2)(3x)' - (3x)(1+x^2)'}{(1+x^2)^2} \\ &= \boxed{\frac{3(1+x^2) - (3x)(2x)}{(1+x^2)^2}} \end{aligned}$$

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Problem 9 Solution

9. The table below shows values of the functions $f(x)$, $g(x)$, and $h(x)$ for x near 0. Based on the data is $h = f'$ or is $h = g'$? Explain your answer by citing some feature of the data.

x	-0.2	-0.1	0	0.1	0.2
$f(x)$	0.494	0.498	0.500	0.498	0.494
$g(x)$	0.460	0.480	0.500	0.519	0.539
$h(x)$	0.059	0.029	0	-0.029	-0.059

Solution: To estimate the derivative $f'(0)$ we use the formula:

$$f'(x) \approx \frac{f(x) - f(2)}{x - 2}$$

Choosing $x = 0.1$ we get the estimate:

$$f'(0) \approx \frac{f(0.1) - f(0)}{0.1 - 0} = \frac{0.498 - 0.500}{0.1} = -0.02$$

Choosing $x = -0.1$ we get the estimate:

$$f'(0) \approx \frac{f(-0.1) - f(0)}{-0.1 - 0} = \frac{0.498 - 0.500}{-0.1} = 0.02$$

The average of these two estimates is:

$$\text{average estimate of } f'(0) = \frac{-0.02 + 0.02}{2} = 0$$

Noting that $h(0) = 0$, it appears as though $h = f'$.

To confirm, we estimate $g'(0)$ using the same technique. We find that

$$\begin{aligned}
g'(0) &\approx \frac{g(0.1) - g(0)}{0.1 - 0} = \frac{0.519 - 0.500}{0.1} = 0.19 \\
g'(0) &\approx \frac{g(-0.1) - g(0)}{-0.1 - 0} = \frac{0.480 - 0.500}{-0.1} = 0.2 \\
\text{average estimate of } g'(0) &= \frac{0.19 + 0.20}{2} = 0.195
\end{aligned}$$

which is decidedly different from $h(0) = 0$ in comparison.

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Problem 10 Solution

10. Suppose that $f(2) = 3$, $f'(2) = -1$, $g(2) = 5$, and $g'(2) = -2$. Find the derivative of the product $f(x)g(x)$ at $x = 2$.

Solution: Using the Product Rule we have:

$$[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$$

At $x = 2$, the value of the derivative $[f(x)g(x)]'$ is:

$$\begin{aligned} [f(x)g(x)]'\Big|_{x=2} &= f(2)g'(2) + f'(2)g(2) \\ &= (3)(-2) + (-1)(5) \\ &= \boxed{-11} \end{aligned}$$