Math 180, Exam 1, Spring 2010
Problem 1 Solution

1. Evaluate the following limits, or show that they do not exist.

(a) $\displaystyle\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

(b) $\displaystyle\lim_{x\to 1} \frac{x^2-3x+2}{x^3-1}$

(c) $\displaystyle\lim_{x\to -1} \frac{|x+1|}{2x+2}$

Solution:

(a) When substituting $x=9$ into the function $f(x)=\dfrac{x-9}{\sqrt{x}-3}$ we find that

$$\frac{x-9}{\sqrt{x}-3}=\frac{9-9}{\sqrt{9}-3}=\frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by multiplying $f(x)$ by the "conjugate" of the denominator divided by itself.

$$\begin{aligned}\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{x-9}{\sqrt{x}-3}\cdot\frac{\sqrt{x}+3}{\sqrt{x}+3}\\ &= \lim_{x\to 9}\frac{(x-9)(\sqrt{x}+3)}{x-9}\\ &= \lim_{x\to 9}(\sqrt{x}+3)\\ &= \sqrt{9}+3\\ &= \boxed{6}\end{aligned}$$

(b) When substituting $x=1$ into the function $f(x)=\dfrac{x^2-3x+2}{x^3-1}$ we find that

$$\frac{x^2-3x+2}{x^3-1}=\frac{1^2-3(1)+2}{1^3-1}=\frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by factoring.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-3x+2}{x^3-1} &= \lim_{x\to 1}\frac{(x-1)(x-2)}{(x-1)(x^2+x+1)}\\ &= \lim_{x\to 1}\frac{x-2}{x^2+x+1}\\ &= \frac{1-2}{1^2+1+1}\\ &= \boxed{-\frac{1}{3}}\end{aligned}$$

(c) When substituting $x = -1$ into the function $f(x) = \dfrac{|x+1|}{2x+2}$ we find that

$$\frac{|x+1|}{2x+2} = \frac{|-1+1|}{2(-1)+2} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by writing the function as a piecewise-defined function.

$$\begin{aligned} f(x) = \frac{|x+1|}{2x+2} &= \begin{cases} \dfrac{x+1}{2x+2} & \text{if } x \geq -1 \\ \dfrac{-(x+1)}{2x+2} & \text{if } x < -1 \end{cases} \\ &= \begin{cases} \dfrac{x+1}{2(x+1)} & \text{if } x \geq -1 \\ \dfrac{-(x+1)}{2(x+1)} & \text{if } x < -1 \end{cases} \\ &= \begin{cases} \dfrac{1}{2} & \text{if } x \geq -1 \\ \dfrac{-1}{2} & \text{if } x < -1 \end{cases} \end{aligned}$$

In order for the limit to exist, the one-sided limits must be the same. However,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -\frac{1}{2} = -\frac{1}{2}$$
$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} \frac{1}{2} = \frac{1}{2}$$

Therefore, the limit does not exist.

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Problem 2 Solution

2. Find the derivatives of the following functions using the basic rules. Leave your answers in an unsimplified form so that it is clear what method you used.

(a) $x^2 \cos x$

(b) $(x^2 - 3x + 14)^{12}$

(c) $\dfrac{x^2 + e^x}{x^2 - e^x}$

Solution:

(a) Use the Product Rule.

$$\begin{aligned}(x^2 \cos x)' &= x^2(\cos x)' + (x^2)' \cos x \\ &= \boxed{-x^2 \sin x + 2x \cos x}\end{aligned}$$

(b) Use the Chain Rule.

$$\begin{aligned}[(x^2 - 3x + 14)^{12}]' &= 12(x^2 - 3x + 14)^{11}(x^2 - 3x + 14)' \\ &= \boxed{12(x^2 - 3x + 14)^{11}(2x - 3)}\end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned}\left(\frac{x^2 + e^x}{x^2 - e^x}\right)' &= \frac{(x^2 - e^x)(x^2 + e^x)' - (x^2 + e^x)(x^2 - e^x)'}{(x^2 - e^x)^2} \\ &= \boxed{\frac{(x^2 - e^x)(2x + e^x) - (x^2 + e^x)(2x - e^x)}{(x^2 - e^x)^2}}\end{aligned}$$

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Problem 3 Solution

3. For the function $f(x) = \dfrac{1}{x}$ compute

(a) The average rate of change from $x = 3$ to $x = 5$.

(b) The instantaneous rate of change at $x = 4$.

Solution:

(a) The average rate of change formula is:

$$\text{average ROC} = \frac{f(b) - f(a)}{b - a}.$$

Using $f(x) = \dfrac{1}{x}$, $b = 5$, and $a = 3$ we have:

$$\text{average ROC} = \frac{\frac{1}{5} - \frac{1}{3}}{5 - 3} = \boxed{-\frac{1}{15}}$$

(b) The instantaneous rate of change at $x = 4$ is $f'(4)$. The derivative $f'(x)$ is:

$$f'(x) = -\frac{1}{x^2}$$

At $x = 4$ we have:

$$\text{instantaneous ROC} = f'(4) = -\frac{1}{4^2} = \boxed{-\frac{1}{16}}$$

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Problem 4 Solution

4. Use the Intermediate Value Theorem in order to show that the equation

$$x^4 = 2^x$$

has at least one real solution.

Solution: Let $f(x) = x^4 - 2^x$. First we recognize that $f(x)$ is continuous everywhere. Next, we must find an interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite signs. Let's choose $a = -1$ and $b = 0$.

$$f(-1) = (-1)^4 - 2^{-1} = \frac{1}{2}$$
$$f(0) = 0^4 - 2^0 = -1$$

Since $f(-1) > 0$ and $f(0) < 0$, the Intermediate Value Theorem tells us that $f(c) = 0$ for some c in the interval $[-1, 0]$.

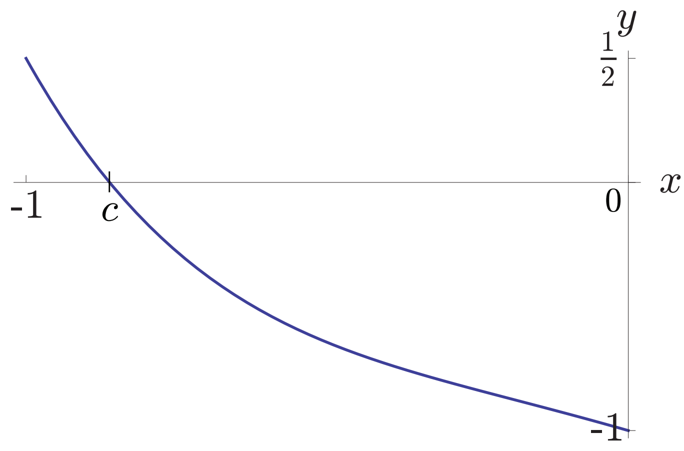

Figure 1: Graph of $f(x) = x^4 - 2^x$ on the interval $[-1, 0]$.

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Problem 5 Solution

5. Find and classify the points of discontinuity of the function

$$\frac{x^2+7x+12}{x^3-9x}.$$

Solution: We start by factoring the numerator and denominator.

$$\frac{x^2+7x+12}{x^3-9x} = \frac{(x+4)(x+3)}{x(x-3)(x+3)}$$

As $x \to 0^+$, we find that:

$$\begin{aligned}\lim_{x\to 0^+} \frac{x^2+7x+12}{x^3-9x} &= \lim_{x\to 0^+} \frac{(x+4)(x+3)}{x(x-3)(x+3)}\\ &= \lim_{x\to 0^+} \frac{x+4}{x(x-3)}\\ &= -\infty\end{aligned}$$

Therefore, $x = 0$ is an infinite discontinuity.

As $x \to 3^+$, we find that:

$$\begin{aligned}\lim_{x\to 3^+} \frac{x^2+7x+12}{x^3-9x} &= \lim_{x\to 0^+} \frac{(x+4)(x+3)}{x(x-3)(x+3)}\\ &= \lim_{x\to 3^+} \frac{x+4}{x(x-3)}\\ &= \infty\end{aligned}$$

Therefore, $x = 3$ is an infinite discontinuity.

The limit at $x = -3$ is:

$$\begin{aligned}\lim_{x\to -3} \frac{x^2+7x+12}{x^3-9x} &= \lim_{x\to -3} \frac{(x+4)(x+3)}{x(x-3)(x+3)}\\ &= \lim_{x\to -3} \frac{x+4}{x(x-3)}\\ &= \frac{-3+4}{(-3)(-3-3)}\\ &= \frac{1}{18}\end{aligned}$$

However, $f(-3)$ does not exist. Using our textbook's definitions, $x = -3$ cannot be categorized as a removable, jump, or infinite discontinuity. Therefore, $x = -3$ falls under the "other" category.

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Problem 6 Solution

6. Find all points where the tangent line to $y = x^3 - 6x + 12$ has slope -1.

Solution: The derivative y' is

$$y' = (x^3 - 6x + 12)' = 3x^2 - 6.$$

To determine the points where the slope of the tangent line is -1, we set the derivative equal to -1 and solve for x.

$$\begin{aligned} y' &= -1 \\ 3x^2 - 6 &= -1 \\ 3x^2 &= 5 \\ x^2 &= \frac{5}{3} \\ \boxed{x = \pm\sqrt{\frac{5}{3}}} \end{aligned}$$