1. Evaluate the following limits, or show that they do not exist

(a) \( \lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} \)

(b) \( \lim_{x \to 3} \frac{|x^2 - 9|}{x^2 + 9} \)

(c) \( \lim_{x \to 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} \)

**Solution:**

(a) Upon substituting \( x = 2 \) we find that

\[
\frac{x^2 + x - 6}{x^2 - 4} = \frac{2^2 + 2 - 6}{2^2 - 4} = \frac{0}{0}
\]

which is indeterminate. To resolve the indeterminacy we factor the numerator and denominator, cancel terms, and evaluate the resulting limit.

\[
\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \frac{(x + 3)(x - 2)}{(x + 2)(x - 2)}
\]

\[
= \lim_{x \to 2} \frac{x + 3}{x + 2}
\]

\[
= \frac{2 + 3}{2 + 2}
\]

\[
= \frac{5}{4}
\]

We were able to substitute \( x = 2 \) after canceling the \( x - 2 \) terms because the function \( \frac{x+3}{x+2} \) is continuous at \( x = 2 \).

(b) Upon substituting \( x = 3 \) we find that:

\[
\lim_{x \to 3} \frac{|x^2 - 9|}{x^2 + 9} = \frac{|3^2 - 9|}{3^2 + 9} = \frac{0}{18} = 0
\]

The substitution method works here because the function \( \frac{|x^2-9|}{x^2+9} \) is continuous everywhere.
(c) Upon substituting \( x = 2 \) we find that

\[
\frac{x - 2}{\sqrt{x} - \sqrt{2}} = \frac{2 - 2}{\sqrt{2} - \sqrt{2}} = \frac{0}{0}
\]

which is indeterminate. To resolve the indeterminacy we multiply the numerator and denominator by \( \sqrt{x} + \sqrt{2} \), cancel terms, and evaluate the resulting limit.

\[
\lim_{x \to 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} = \lim_{x \to 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}}
\]

\[
= \lim_{x \to 2} \frac{(x - 2)(\sqrt{x} + \sqrt{2})}{x - 2}
\]

\[
= \lim_{x \to 2} (\sqrt{x} + \sqrt{2})
\]

\[
= \sqrt{2} + \sqrt{2}
\]

\[
= 2\sqrt{2}
\]

We were able to substitute \( x = 2 \) after canceling the \( x - 2 \) terms because the function \( \sqrt{x} + \sqrt{2} \) is continuous at \( x = 2 \).
2. Consider the equation \(x^3 + x + 1 = 0\).

(a) Use the Intermediate Value Theorem to show that it has a solution in the interval \([-2, 0]\).

(b) Use the Bisection Method to find an interval of length \(\frac{1}{2}\) that contains a solution.

Solution: Let \(f(x) = x^3 + x + 1\).

(a) Since \(f(x)\) is a polynomial, we know that it is continuous everywhere. Furthermore, \(f(0) = 1\) and \(f(-2) = -9\) have opposite signs. Therefore, the Intermediate Value Theorem guarantees the existence of a zero of \(f(x)\) on the interval \([-2, 0]\).

(b) The midpoint of \([-2, 0]\) is \(x = -1\). Since \(f(-1) = -1\) and \(f(0) = 1\) have opposite signs, we know that a zero of \(f(x)\) exists in the interval \([-1, 0]\).

The midpoint of \([-1, 0]\) is \(x = -\frac{1}{2}\). Since \(f(-\frac{1}{2}) = \frac{3}{8}\) and \(f(-1) = -1\) have opposite signs, we know that a zero of \(f(x)\) exists in the interval \([-1, -\frac{1}{2}]\). This interval has a length of \(\frac{1}{2}\) so we’re done.
3. Let \( f(x) = x^2 - 2x - 2 \).

(a) Use the definition of the derivative as a limit of difference quotients to compute \( f'(3) \).

(b) Find an equation of the tangent line to the graph of \( f \) at the point \((3, 1)\).

Solution:

(a) Using the limit definition of the derivative, we can compute \( f'(3) \) using either of the following formulas:

\[
\begin{align*}
f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}
\end{align*}
\]

Using the first formula, we get:

\[
\begin{align*}
f'(3) &= \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{x \to 3} \frac{(x^2 - 2x - 2) - (3^2 - 2(3) - 2)}{x - 3} \\
&= \lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} \\
&= \lim_{x \to 3} \frac{(x + 1)(x - 3)}{x - 3} \\
&= \lim_{x \to 3} (x + 1) \\
&= 3 + 1 \\
&= 4
\end{align*}
\]

(b) Using the fact that \( f'(3) = 4 \) is the slope of the tangent line at the point \((3, 1)\), an equation for the tangent line in point-slope form is:

\[
y - 1 = 4(x - 3)
\]
4. Let $f(x) = \frac{1}{1-x} + 3$.

(a) Find the average rate of change of the function between $x = -0.6$ and $x = -0.4$.

(b) Find the instantaneous rate of change at $x = -0.5$.

Solution:

(a) The average rate of change of $f(x)$ on the interval $[-0.6, -0.4]$ is:

$$\text{average ROC} = \frac{f(-0.4) - f(-0.6)}{-0.4 - (-0.6)} = \frac{\left(\frac{1}{1-(-0.4)} + 3\right) - \left(\frac{1}{1-(-0.6)} + 3\right)}{-0.4 - (-0.6)}$$

$$= \frac{\frac{1}{1.4} - \frac{1}{1.6}}{0.2} = \frac{25}{56}$$

(b) The instantaneous rate of change at $x = -0.5$ is $f'(-0.5)$. The derivative $f'(x)$ is found using the Chain Rule.

$$f'(x) = \left(\frac{1}{1-x} + 3\right)'$$

$$= -(1-x)^{-2} \cdot (1-x)' + 3'$$

$$= \frac{1}{(1-x)^2}$$

At $x = -0.5$, we have:

$$f'(-0.5) = \frac{1}{(1 - (-0.5))^2} = \frac{4}{9}$$
5. Find the derivatives of the following functions using the basic rules. Leave your answers in an unsimplified form so that it is clear what method you used.

(a) \( f(x) = \sin(x^3) \)

(b) \( f(x) = x^2 \cdot \arctan(3x) \)

(c) \( f(x) = \frac{1 - \cos x}{x^2 + 1} \)

(d) \( f(x) = x^3e^{-x} \).

Solution:

(a) Use the Chain Rule.

\[
 f'(x) = [\sin(x^3)]' \\
 = \cos(x^3) \cdot (x^3)' \\
 = \cos(x^3) \cdot 3x^2
\]

(b) Use the Product and Chain Rules.

\[
 f'(x) = [x^2 \cdot \arctan(3x)]' \\
 = x^2 \cdot [\arctan(3x)]' + (x^2)' \cdot \arctan(3x) \\
 = x^2 \cdot \frac{1}{1+(3x)^2} \cdot (3x)' + 2x \cdot \arctan(3x) \\
 = x^2 \cdot \frac{1}{1+(3x)^2} \cdot 3 + 2x \cdot \arctan(3x)
\]

(c) Use the Quotient Rule.

\[
 f'(x) = \left(\frac{1 - \cos x}{x^2 + 1}\right)' \\
 = \frac{(x^2 + 1)(1 - \cos x)' - (1 - \cos x)(x^2 + 1)'}{(x^2 + 1)^2} \\
 = \frac{(x^2 + 1)(\sin x) - (1 - \cos x)(2x)}{(x^2 + 1)^2}
\]
(d) Use the Product and Chain Rules.

\[
\begin{align*}
  f'(x) &= (x^3 e^{-x})' \\
  &= x^3 (e^{-x})' + e^{-x} (x^3)' \\
  &= x^3 (-e^{-x}) + e^{-x} (3x^2)
\end{align*}
\]
6. Find the value/s of $c$ for which the function

$$f(x) = \begin{cases} 
  x^2 + 3 & \text{if } x < 2 \\
  cx - 1 & \text{if } x \geq 2 
\end{cases}$$

is continuous at $x = 2$. Justify your answers.

**Solution:** $f(x)$ will be continuous at $x = 2$ if

$$\lim_{x \to 2^-} f(x) = f(2)$$

To determine the limit, we must consider the one-sided limits as $x \to 2$. The limit as $x \to 2^-$ is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 3) = 2^2 + 3 = 7$$

The limit as $x \to 2^+$ is

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (cx - 1) = c(2) - 1 = 2c - 1$$

In order for the limit to exist, the one-sided limits must be the same. So we must have:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

$$2c - 1 = 7$$

$$c = 4$$

Thus, when $c = 4$ the one-sided limits are the same and both are equal to 7. Furthermore, when $c = 4$ we know that $f(2) = 4(2) - 1 = 7$, so the function is continuous at $x = 2$ when $c = 4$. 