Math 180, Exam 2, Fall 2010
Problem 1 Solution

1. Find the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{e^x - 1}{\sin x}$

(b) $\displaystyle\lim_{x\to 0} \frac{e^x}{\sin x - 1}$

(c) $\displaystyle\lim_{x\to \infty} \frac{4x^3 - 3x + 8}{6x^3 + x^2 + x - 12}$

Solution:

(a) Upon substituting $x = 0$ into the function $f(x) = \dfrac{e^x - 1}{\sin x}$ we find that

$$\frac{e^x - 1}{\sin x} = \frac{e^0 - 1}{\sin 0} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy by using L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1}{\sin x} &\overset{\text{L'H}}{=} \lim_{x\to 0} \frac{(e^x - 1)'}{(\sin x)'} \\ &= \lim_{x\to 0} \frac{e^x}{\cos x} \\ &= \frac{e^0}{\cos 0} \\ &= \boxed{1}\end{aligned}$$

(b) The function $f(x) = \dfrac{e^x}{\sin x - 1}$ is continuous at $x = 0$. Therefore, we can find the value of the limit using the substitution method.

$$\lim_{x\to 0} \frac{e^x}{\sin x - 1} = \frac{e^0}{\sin 0 - 1} = \boxed{-1}$$

(c) Since the degrees of the numerator and denominator of the rational function $f(x) = \dfrac{4x^3 - 3x + 8}{6x^3 + x^2 + x - 12}$ are the same, the limit of $f(x)$ as $x \to \infty$ is the ratio of the leading coefficients. That is,

$$\lim_{x\to \infty} \frac{4x^3 - 3x + 8}{6x^3 + x^2 + x - 12} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

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Problem 2 Solution

2. Let $f(x) = \dfrac{x^2}{x+1}$.

(a) Find the critical points of f.

(b) Use the second derivative test to classify the critical points as maxima or minima.

(c) Find the absolute minimum and maximum values of f on the interval $[-3, -\frac{3}{2}]$.

Solution:

(a) The critical points of $f(x)$ are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. The derivative $f'(x)$ can be found using the quotient rule.

$$\begin{aligned} f'(x) &= \left(\frac{x^2}{x+1}\right)' \\ &= \frac{(x+1)(x^2)' - (x^2)(x+1)'}{(x+1)^2} \\ &= \frac{(x+1)(2x) - (x^2)(1)}{(x+1)^2} \\ &= \frac{x^2+2x}{(x+1)^2} \end{aligned}$$

$f'(x)$ exists for all $x \neq -1$ but $x = -1$ is not in the domain of f. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{x^2+2x}{(x+1)^2} &= 0 \\ x^2 + 2x &= 0 \\ x(x+2) &= 0 \\ x = 0,\ x &= -2 \end{aligned}$$

The corresponding function values are $f(0) = 0$ and $f(-2) = -4$. Thus, the critical points are $(0, 0)$ and $(-2, -4)$.

(b) We use the Second Derivative Test to classify the critical points. The second derivative is found using the quotient rule.

$$\begin{aligned}
f''(x) &= \left(\frac{x^2+2x}{(x+1)^2}\right)' \\
&= \frac{(x+1)^2(x^2+2x)' - (x^2+2x)[(x+1)^2]'}{[(x+1)^2]^2} \\
&= \frac{(x+1)^2(2x+2) - (x^2+2x)[2(x+1)]}{(x+1)^4} \\
&= \frac{2(x+1)^3 - 2x(x+2)(x+1)}{(x+1)^4} \\
&= \frac{2(x+1)^2 - 2x(x+2)}{(x+1)^3} \\
&= \frac{2x^2+4x+2-2x^2-4x}{(x+1)^3} \\
&= \frac{2}{(x+1)^3}
\end{aligned}$$

At the critical points, we have:

$$\begin{aligned}
f''(0) &= \frac{2}{(0+1)^3} = 2 \\
f''(-2) &= \frac{2}{(-2+1)^3} = -2
\end{aligned}$$

Since $f''(-2) < 0$ the Second Derivative Test implies that $f(-2) = -4$ is a local maximum. Since $f''(0) > 0$ the Second Derivative Test implies that $f(0) = 0$ is a local minimum.

(c) The absolute extrema of f will occur either at a critical point in $[-3, -\frac{3}{2}]$ or at one of the endpoints. From part (a), we found that the critical numbers of f are $x = 0$ and $x = -2$. Since $x = 0$ is outside the interval, we only evaluate f at $x = -3$, $x = -2$, and $x = -\frac{3}{2}$.

$$\begin{aligned}
f(-3) &= \frac{(-3)^2}{-3+1} = -\frac{9}{2} \\
f(-2) &= \frac{(-2)^2}{-2+1} = -4 \\
f(-\tfrac{3}{2}) &= \frac{(-\frac{3}{2})^2}{-\frac{3}{2}+1} = -\frac{9}{2}
\end{aligned}$$

The absolute maximum of f on $[-3, -\frac{3}{2}]$ is $\boxed{-4}$ because it is the largest of the values of f above and the absolute minimum is $\boxed{-\frac{9}{2}}$ because it is the smallest.

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Problem 3 Solution

3. Estimate a root of the polynomial $f(x) = x^3 + x + 3 = 0$ by performing one step of Newton's method, beginning with $x_0 = -1$.

Solution: The Newton's method formula to compute x_1 is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

where $f(x) = x^3 + x + 3$. The derivative $f'(x)$ is $f'(x) = 3x^2 + 1$. Plugging $x_0 = -1$ into the formula we get:

$$\begin{aligned}
x_1 &= x_0 - \frac{x_0^3 + x_0 + 3}{3x_0^2 + 1} \\
x_1 &= -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} \\
x_1 &= -1 - \frac{-1 - 1 + 3}{3 + 1} \\
&\boxed{x_1 = -\frac{5}{4}}
\end{aligned}$$

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Problem 4 Solution

4. Find the point on the parabola $y = x^2$ which is closest to the point $(3, 0)$.

Solution: The function we seek to minimize is the distance between (x, y) and $(3, 0)$.

$$\textbf{Function}: \qquad \text{Distance} = \sqrt{(x-3)^2 + (y-0)^2} \tag{1}$$

The constraint in this problem is that the point (x, y) must lie on the curve $y = x^2$.

$$\textbf{Constraint}: \qquad y = x^2 \tag{2}$$

Plugging this into the distance function (1) and simplifying we get:

$$\begin{aligned} \text{Distance} &= \sqrt{(x-3)^2 + (x^2-0)^2} \\ f(x) &= \sqrt{x^2 - 6x + 9 + x^4} \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(-\infty, \infty)$. We choose this interval because (x, y) must be on the parabola $y = x^2$ and the domain of this function is $(-\infty, \infty)$.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(-\infty, \infty)$ or it will not exist. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left[\left(x^2 - 6x + 9 + x^4\right)^{1/2}\right]' &= 0 \\ \frac{1}{2}\left(x^2 - 6x + 9 + x^4\right)^{-1/2} \cdot \left(x^2 - 6x + 9 + x^4\right)' &= 0 \\ \frac{2x - 6 + 4x^3}{2\sqrt{x^2 - 6x + 9 + x^4}} &= 0 \\ 2x - 6 + 4x^3 &= 0 \\ 2x^3 + x - 3 &= 0 \end{aligned}$$

This is a cubic equation, which is slightly difficult to solve. However, by inspection, we notice that $x = 1$ is a solution (the other two solutions are complex). Plugging this into $f(x)$ we get:

$$f(1) = \sqrt{1^2 - 6(1) + 9 + 1^4} = \sqrt{5}$$

Taking the limit as $x \to \pm\infty$ we get:

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \sqrt{x^2 - 6x + 9 + x^4} = \infty \\ \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \sqrt{x^2 - 6x + 9 + x^4} = \infty \end{aligned}$$

both of which are larger than $\sqrt{5}$. We conclude that the distance is an absolute minimum at $\boxed{x = 1}$ and that the resulting distance is $\sqrt{5}$. The last step is to find the corresponding value for y by plugging $x = 1$ into equation (2).

$$\boxed{y = 1^2 = 1}$$

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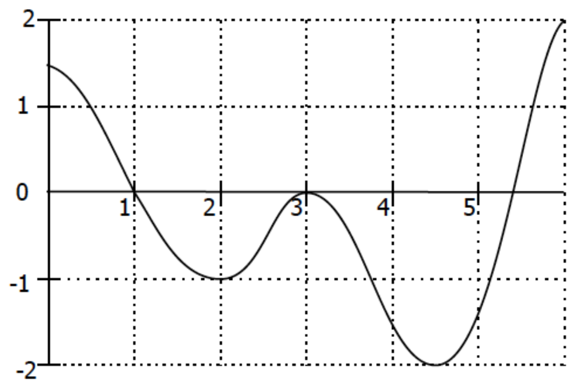
Problem 5 Solution

5. Shown below is the graph of $f'(x)$, the **derivative** of the function $f(x)$.

(a) Using the graph of $f'(x)$ below, determine the intervals where $f(x)$ is increasing, decreasing, concave up, or concave down.

(b) Given that $f(0) = 0$, use your results from part (a) to sketch the graph of $f(x)$ for $x \in [0, 6]$.

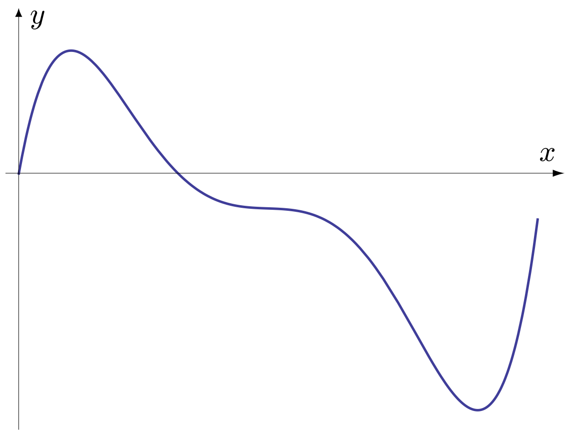
(c) On the graph of $f(x)$ that you sketched in part (b), clearly label all maxima, minima, and inflection points.



Solution:

(a) $f(x)$ is increasing on (a, b) when $f'(x) > 0$ for all $x \in (a, b)$. This occurs on $(0, 1) \cup (5.5, 6)$ because the graph is above the x-axis for these values of x. $f(x)$ is decreasing on (a, b) when $f'(x) < 0$ for all $x \in (a, b)$. This occurs on $(1, 3) \cup (3, 5.5)$ because the graph is below the x-axis for these values of x. $f(x)$ is concave up on (a, b) when $f'(x)$ is increasing on (a, b). This occurs on $(2, 3) \cup (4.5, 6)$ because the graph is rising for these values of x. $f(x)$ is concave down on (a, b) when $f'(x)$ is decreasing on (a, b). This occurs on $(0, 2) \cup (3, 4.5)$ because the graph is falling for these values of x.

(b) The general shape of the graph is shown below. (Note: the graph is not necessarily to scale.)



(c) f has a local maximum at $x = c$ when $f'(c) = 0$ and the sign of f' changes from positive to negative at $x = c$. This occurs at $x = 1$. f has a local minimum at $x = c$ when $f'(c) = 0$ and the sign of f' changes from negative to positive at $x = c$. This occurs at $x = 5.5$. f has an inflection point at $x = c$ when $f''(c) = 0$ and the sign of f'' changes at $x = c$. This occurs at $x = 2, 3, 4.5$.