Math 180, Exam 2, Fall 2011
Problem 1 Solution

1. Find the derivative of each function. Do not simplify your answers.

(a) $\log_5(6 + \sin x)$

(b) $x^{\sin x}$

(c) $\tan^{-1}(e^{1-x})$

Solution:

(a) Use the Chain Rule.

$$\frac{d}{dx}\log_5(6+\sin x) = \frac{1}{\ln 5}\cdot\frac{1}{6+\sin x}\cdot(6+\sin x)'$$
$$= \boxed{\frac{1}{\ln 5}\cdot\frac{1}{6+\sin x}\cdot\cos x}$$

(b) First rewrite the function as the exponential of a logarithm and simplify using a logarithm rule.

$$x^{\sin x} = e^{\ln x^{\sin x}} = e^{\sin x \ln x}$$

Now use the Chain and Product Rules.

$$\frac{d}{dx}x^{\sin x} = \frac{d}{dx}e^{\sin x\ln x}$$
$$= e^{\sin x\ln x}\left[(\sin x)(\ln x)' + (\sin x)'(\ln x)\right]$$
$$= e^{\sin x\ln x}\left(\frac{\sin x}{x} + \cos x\ln x\right)$$
$$= \boxed{x^{\sin x}\left(\frac{\sin x}{x} + \cos x\ln x\right)}$$

(c) Use the Chain Rule.

$$\frac{d}{dx}\tan^{-1}(e^{1-x}) = \frac{1}{1+(e^{1-x})^2}\cdot(e^{1-x})'$$
$$= \boxed{\frac{1}{1+(e^{1-x})^2}\cdot\left(-e^{1-x}\right)}$$

Math 180, Exam 2, Fall 2011
Problem 2 Solution

2. Find a point (x, y) on the graph of $y = \dfrac{x^2}{6} + 4$ nearest the point $P = (0, 13)$.
Hint: Find the minimum value of the square of the distance between (x, y) and P.

Solution: The function we seek to minimize is the square of the distance between (x, y) and $(0, 13)$.

$$\textbf{Function}: \qquad \text{Distance}^2 = (x-0)^2 + (y-13)^2 \tag{1}$$

The constraint in this problem is that the point (x, y) must lie on the curve $y = \dfrac{x^2}{6} + 4$.

$$\textbf{Constraint}: \qquad y = \frac{x^2}{6} + 4 \tag{2}$$

Plugging this into the distance function (1) and simplifying we get:

$$\begin{aligned} \text{Distance}^2 &= (x-0)^2 + \left(\frac{x^2}{6} + 4 - 13\right)^2 \\ f(x) &= x^2 + \left(\frac{x^2}{6} - 9\right)^2 \\ f(x) &= x^2 + \frac{x^4}{36} - 3x^2 + 81 \\ f(x) &= \frac{x^4}{36} - 2x^2 + 81 \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(-\infty, \infty)$. We choose this interval because (x, y) must be on the parabola $y = \frac{x^2}{6} + 4$ and the domain of this function is $(-\infty, \infty)$.

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(-\infty, \infty)$ or it will not exist. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{d}{dx}\left(\frac{x^4}{36} - 2x^2 + 81\right) &= 0 \\ \frac{x^3}{9} - 4x &= 0 \\ x\left(\frac{x^2}{9} - 4\right) &= 0 \\ x = 0 \quad \text{or} \quad \frac{x^2}{9} &= 4 \\ x = 0 \quad \text{or} \quad x &= \pm 6 \end{aligned}$$

Plugging these values into $f(x)$ we get:

$$f(0) = \frac{0^4}{36} - 2(0)^2 + 81 = 81$$
$$f(-6) = \frac{(-6)^4}{36} - 2(-6)^2 + 81 = 45$$
$$f(6) = \frac{6^4}{36} - 2(6)^2 + 81 = 45$$

Taking the limit as $x \to \pm\infty$ we get:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(\frac{x^4}{36} - 2x^2 + 81\right) = \infty$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(\frac{x^4}{36} - 2x^2 + 81\right) = \infty.$$

The smallest of the above values of the function and of the limits is 45. Thus, we conclude that the distance is an absolute minimum at $\boxed{x = \pm 6}$ and that the resulting square of the distance is 45. The last step is to find the corresponding values for y by plugging $x = \pm 6$ into equation (2).

$$\boxed{y = \frac{(\pm 6)^2}{6} + 4 = 10}$$

Math 180, Exam 2, Fall 2011
Problem 3 Solution

3. Consider the equation $x^2 + xy + 2y^2 = 4$.

(a) Use implicit differentiation to compute the derivative $\dfrac{dy}{dx}$.

(b) Find an equation for the tangent line to the curve at $(1, 1)$.

Solution:

(a) Using implicit differentiation we get:

$$\begin{aligned} \frac{d}{dx} x^2 + \frac{d}{dx}(xy) + \frac{d}{dx} 2y^2 &= \frac{d}{dx} 4 \\ 2x + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} + 4y\frac{dy}{dx} &= -2x - y \\ \frac{dy}{dx}(x + 4y) &= -2x - y \\ \boxed{\frac{dy}{dx} = \frac{-2x - y}{x + 4y}} \end{aligned}$$

(b) At the point $(1, 1)$, the value of $\dfrac{dy}{dx}$ is:

$$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{-2(1) - 1}{1 + 4(1)} = -\frac{3}{5}$$

This is the slope of the tangent line at $(1, 1)$. Therefore, an equation for the tangent line in point-slope form is

$$\boxed{y - 1 = -\frac{3}{5}(x - 1).}$$

Math 180, Exam 2, Fall 2011
Problem 4 Solution

4. (a) Verify that $f(x) = x\sqrt{x+6}$ satisfies the hypotheses of Rolle's Theorem on the interval $[-6, 0]$.

(b) Find all numbers c that satisfy the conclusion of Rolle's Theorem.

Solution:

(a) First, we note that $f(x) = x\sqrt{x+6}$ is continuous on $[-6, 0]$. Next, the derivative $f'(x)$ is

$$f'(x) = \sqrt{x+6} + \frac{x}{2\sqrt{x+6}}$$

which exists for all x in $(-6, 0)$. Finally, we have $f(-6) = f(0) = 0$. Therefore, Rolle's Theorem can be applied.

(b) The conclusion of Rolle's Theorem is that there exists at least one c in $(-6, 0)$ such that $f'(c) = 0$. The corresponding value of c are

$$\begin{aligned} f'(c) &= 0, \\ \sqrt{c+6} + \frac{c}{2\sqrt{c+6}} &= 0, \\ 2(c+6) + c &= 0, \\ 3c &= -12, \\ \boxed{c = -4} \end{aligned}$$

Math 180, Exam 2, Fall 2011
Problem 5 Solution

5. Consider the function $f(x) = x^4 - 2x^2$.

(a) Find the intervals on which f is increasing or decreasing.

(b) Find the intervals on which f is concave up or concave down.

(c) Find the local extrema of f. Which, if any, are absolute extrema?

Solution:

(a) We begin by finding the critical points of $f(x)$. These occur when either $f'(x)$ does not exist or $f'(x) = 0$. Since $f(x)$ is a polynomial we know that $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^4 - 2x^2\right)' &= 0 \\ 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, \ x = \pm 1. \end{aligned}$$

The domain of $f(x)$ is $(-\infty, \infty)$. We now split the domain into the four intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, -1)$	-2	$f'(-2) = -24$	$-$
$(-1, 0)$	$-\frac{1}{2}$	$f'(-\frac{1}{2}) = \frac{3}{2}$	$+$
$(0, 1)$	$\frac{1}{2}$	$f'(\frac{1}{2}) = -\frac{3}{2}$	$-$
$(1, \infty)$	2	$f'(2) = 24$	$+$

Using the table we conclude that $f(x)$ is increasing on $(-1, 0) \cup (1, \infty)$ because $f'(x) > 0$ for all $x \in (-1, 0) \cup (1, \infty)$ and $f(x)$ is decreasing on $(-\infty, -1) \cup (0, 1)$ because $f'(x) < 0$ for all $x \in (-\infty, -1) \cup (0, 1)$.

(b) To find the intervals of concavity we begin by finding solutions to $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 0 \\ \left(4x^3 - 4x\right)' &= 0 \\ 12x^2 - 4 &= 0 \\ x^2 &= \frac{1}{3} \\ x &= \pm \frac{1}{\sqrt{3}} \end{aligned}$$

We now split the domain into the three intervals $(-\infty, -\frac{1}{\sqrt{3}})$, $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and $(\frac{1}{\sqrt{3}}, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, -\frac{1}{\sqrt{3}})$	-1	$f''(-1) = 8$	$+$
$(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	0	$f''(0) = -4$	$-$
$(\frac{1}{\sqrt{3}}, \infty)$	1	$f''(1) = 8$	$+$

Using the table we conclude that $f(x)$ is concave up on $(-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ because $f''(x) > 0$ for all $x \in (-\infty, -\frac{1}{\sqrt{3}}) \cup (\frac{1}{\sqrt{3}}, \infty)$ and $f(x)$ is concave down on $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ because $f''(x) < 0$ for all $x \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

(c) Using the First Derivative Test and the table from part (a), we conclude that the points $(-1, -1)$ and $(1, -1)$ correspond to local minima and the point $(0, 0)$ corresponds to a local maximum. Furthermore, since

$$\lim_{x \to \pm\infty} f(x) = +\infty$$

we know that $(-1, -1)$ and $(1, -1)$ also correspond to absolute minima. However, $f(x)$ has no absolute maximum on its domain.