Math 180, Final Exam, Spring 2008
Problem 1 Solution

1. For each of the following limits, determine whether the limit exists and, if so, evaluate it.

(a) $\lim_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{x}$

(b) $\lim_{x\to 1} \dfrac{|(x-1)^3|}{x-1}$

(c) $\lim_{x\to 1} \dfrac{|x-1|}{x-1}$

Solution:

(a) Upon substituting $x = 0$ into the function $f(x) = \dfrac{\sqrt{x^2+1}-1}{x}$ we find that

$$\frac{\sqrt{x^2+1}-1}{x} = \frac{\sqrt{0^2+1}-1}{0} = \frac{0}{0}$$

which is indeterminate. We can resolve the indeterminacy by multiplying $f(x)$ by the "conjugate" of the numerator divided by itself.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{x} \cdot \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \\
&= \lim_{x\to 0} \frac{(x^2+1)-1}{x(\sqrt{x^2+1}+1)} \\
&= \lim_{x\to 0} \frac{x^2}{x(\sqrt{x^2+1}+1)} \\
&= \lim_{x\to 0} \frac{x}{\sqrt{x^2+1}+1} \\
&= \frac{0}{\sqrt{0^2+1}+1} \\
&= \boxed{0}
\end{aligned}$$

(b) When substituting $x = 1$ into the function $f(x) = \dfrac{|(x-1)^3|}{x-1}$ we find that

$$\frac{|(x-1)^3|}{x-1} = \frac{|(1-1)^3|}{1-1} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy by computing the one-sided limits.

$$\begin{aligned}
\lim_{x\to 1^-} \frac{|(x-1)^3|}{x-1} &= \lim_{x\to 1^-} \frac{-(x-1)^3}{x-1} \\
&= \lim_{x\to 1^-} -(x-1)^2 \\
&= -(1-1)^2 \\
&= 0 \\
\lim_{x\to 1^+} \frac{|(x-1)^3|}{x-1} &= \lim_{x\to 1^+} \frac{(x-1)^3}{x-1} \\
&= \lim_{x\to 1^+} (x-1)^2 \\
&= (1-1)^2 \\
&= 0
\end{aligned}$$

Thus, since the one-sided limits are equal to each other, we know that the limit exists and that:

$$\lim_{x\to 1} \frac{|(x-1)^3|}{x-1} = \boxed{0}$$

(c) When substituting $x=1$ into the function $f(x) = \dfrac{|x-1|}{x-1}$ we find that

$$\frac{|x-1|}{x-1} = \frac{|1-1|}{1-1} = \frac{0}{0}$$

which is indeterminate. We resolve the indeterminacy by computing the one-sided limits.

$$\begin{aligned}
\lim_{x\to 1^-} \frac{|x-1|}{x-1} &= \lim_{x\to 1^-} \frac{-(x-1)}{x-1} \\
&= \lim_{x\to 1^-} -1 \\
&= -1 \\
\lim_{x\to 1^+} \frac{|x-1|}{x-1} &= \lim_{x\to 1^+} \frac{x-1}{x-1} \\
&= \lim_{x\to 1^+} 1 \\
&= 1
\end{aligned}$$

Thus, since the one-sided limits are not equal to each other, we know that the limit does not exist.

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Problem 2 Solution

2. Find the derivatives of the following functions. Show each step and do not simplify your answer.

(a) $f(x) = x^{99} - 23x^2 + \cos x$

(b) $g(t) = t^2 e^{1/t}$

(c) $h(x) = \dfrac{x + \cos x}{x - \sin x}$

(d) $k(t) = \ln(1 + \sqrt{t^2 + 1})$

Solution:

(a) Use the Power Rule and basic derivative of $\cos x$.

$$f'(x) = (x^{99} - 23x^2 + \cos x)' = \boxed{99x^{98} - 46x - \sin x}$$

(b) Use the Product and Chain Rules.

$$\begin{aligned} g'(t) &= \left(t^2 e^{1/t}\right)' \\ &= (t^2)(e^{1/t})' + (t^2)'(e^{1/t}) \\ &= t^2 e^{1/t} \cdot \left(\frac{1}{t}\right)' + 2te^{1/t} \\ &= \boxed{t^2 e^{1/t} \cdot \left(-\frac{1}{t^2}\right) + 2te^{1/t}} \end{aligned}$$

(c) Use the Quotient Rule.

$$\begin{aligned} h'(x) &= \left(\frac{x + \cos x}{x - \sin x}\right)' \\ &= \frac{(x - \sin x)(x + \cos x)' - (x + \cos x)(x - \sin x)'}{(x - \sin x)^2} \\ &= \boxed{\frac{(x - \sin x)(1 - \sin x) - (x + \cos x)(1 - \cos x)}{(x - \sin x)^2}} \end{aligned}$$

(d) Use the Chain Rule.

$$
\begin{aligned}
k'(t) &= \left[\ln(1+\sqrt{t^2+1})\right]' \\
&= \frac{1}{1+\sqrt{t^2+1}} \cdot \left(1+\sqrt{t^2+1}\right)' \\
&= \frac{1}{1+\sqrt{t^2+1}} \cdot \left(\frac{1}{2}(t^2+1)^{-1/2} \cdot (t^2+1)'\right) \\
&= \boxed{\frac{1}{1+\sqrt{t^2+1}} \cdot \left(\frac{1}{2}(t^2+1)^{-1/2} \cdot (2t)\right)}
\end{aligned}
$$

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Problem 3 Solution

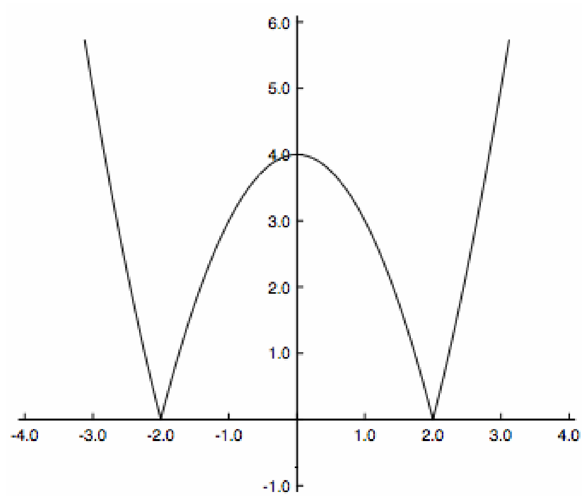
3. The graph of $f(x)$ is shown below.

(a) For which values of x does $f'(x)$ not exist?

(b) For which values of x is $f'(x) = 0$?

(c) For which values of x is $f'(x) > 0$?

(d) Sketch the graph of $f'(x)$ for $-4 \leq x \leq 4$.



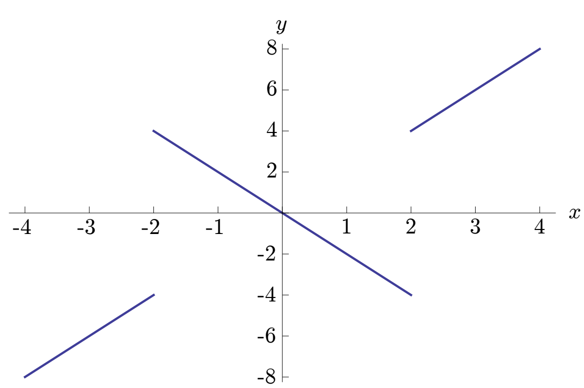
Solution:

(a) $f'(x)$ does not exist at $x = -2$ and $x = 2$.

(b) $f'(x) = 0$ at $x = 0$

(c) $f'(x) > 0$ on $(-2, 0) \cup (2, \infty)$

(d) The graph of $y = f'(x)$ is below:



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Problem 4 Solution

4. Evaluate the following definite integrals:

(a) $\displaystyle\int_1^2 x^3 - 3x^2 \, dx \, dx$

(b) $\displaystyle\int_1^4 x\sqrt{1+x^2} \, dx$

(c) $\displaystyle\int_0^{\pi/2} \sin(x) \, dx$

Solution:

(a) We use the Fundamental Theorem of Calculus, Part I.

$$\begin{aligned}
\int_1^2 x^3 - 3x^2 \, dx &= \frac{x^4}{4} - x^3 \Big|_1^2 \\
&= \left(\frac{2^4}{4} - 2^3\right) - \left(\frac{1^4}{4} - 1^3\right) \\
&= (4 - 8) - \left(\frac{1}{4} - 1\right) \\
&= \boxed{-\frac{13}{4}}
\end{aligned}$$

(b) We use the substitution $u = 1 + x^2$, $\frac{1}{2}\,du = x\,dx$. The limits of integration become $u = 1 + 1^2 = 2$ and $u = 1 + 4^2 = 17$. Making the substitutions and using the Fundamental Theorem of Calculus, Part I the integral is then:

$$\begin{aligned}
\int_1^4 x\sqrt{1+x^2} \, dx &= \frac{1}{2}\int_2^{17} \sqrt{u} \, du \\
&= \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_2^{17} \\
&= \frac{1}{2}\left[\frac{2}{3}17^{3/2} - \frac{2}{3}2^{3/2}\right] \\
&= \boxed{\frac{1}{3}\left(17\sqrt{17} - 2\sqrt{2}\right)}
\end{aligned}$$

(c) We use the Fundamental Theorem of Calculus, Part I.

$$\begin{aligned}
\int_0^{\pi/2} \sin(x)\,dx &= -\cos(x)\Big|_0^{\pi/2} \\
&= -\cos\left(\frac{\pi}{2}\right) - (-\cos 0) \\
&= -0 - (-1) \\
&= \boxed{1}
\end{aligned}$$

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Problem 5 Solution

5. Find an equation for the tangent line to the curve given by the equation

$$x^3 + y^3 = 6xy$$

at the point $(\frac{4}{3}, \frac{8}{3})$.

Solution: We must find $\frac{dy}{dx}$ using implicit differentiation.

$$\begin{aligned}
x^3 + y^3 &= 6xy \\
\frac{d}{dx}x^3 + \frac{d}{dx}y^3 &= \frac{d}{dx}(6xy) \\
3x^2 + 3y^2\frac{dy}{dx} &= 6x\frac{dy}{dx} + 6y \\
3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} &= -3x^2 + 6y \\
\frac{dy}{dx}(3y^2 - 6x) &= -3x^2 + 6y \\
\frac{dy}{dx} &= \frac{-3x^2 + 6y}{3y^2 - 6x} \\
\frac{dy}{dx} &= \frac{x^2 - 2y}{-y^2 + 2x}
\end{aligned}$$

The value of $\frac{dy}{dx}$ at $(\frac{4}{3}, \frac{8}{3})$ is the slope of the tangent line.

$$\left.\frac{dy}{dx}\right|_{(\frac{4}{3}, \frac{8}{3})} = \frac{(\frac{4}{3})^2 - 2(\frac{8}{3})}{-(\frac{8}{3})^2 + 2(\frac{4}{3})} = \frac{4}{5}$$

An equation for the tangent line at $(\frac{4}{3}, \frac{8}{3})$ is then:

$$\boxed{y - \frac{8}{3} = \frac{4}{5}\left(x - \frac{4}{3}\right)}$$

Math 180, Exam 2, Spring 2008
Problem 6 Solution

6. Find the maximum and minimum values for the function $f(x) = x^3 - 3x^2$ on the interval $[-1, 1]$.

Solution: The minimum and maximum values of $f(x)$ will occur at a critical point in the interval $[-1, 1]$ or at one of the endpoints. The critical points are the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist. Since $f(x)$ is a polynomial, $f'(x)$ exists for all $x \in \mathbb{R}$. Therefore, the only critical points are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x^3 - 3x^2\right)' &= 0 \\ 3x^3 - 6x &= 0 \\ 3x(x^2 - 1) &= 0 \\ 3x(x-1)(x+1) &= 0 \\ x = 0,\ x &= \pm 1 \end{aligned}$$

The critical points $x = 0$ lies in the interior of $[-1, 1]$ while $x = \pm 1$ are the endpoints of the interval. Therefore, we check the value of $f(x)$ at $x = -1,\ 0,\ \text{and } 1$.

$$\begin{aligned} f(-1) &= (-1)^3 - 3(-1)^2 = -4 \\ f(0) &= 0^3 - 3(0)^2 = 0 \\ f(1) &= 1^3 - 3(1)^2 = -2 \end{aligned}$$

The minimum value of $f(x)$ on $[-1, 1]$ is $\boxed{-4}$ because it is the smallest of the above values of f. The maximum is $\boxed{0}$ because it is the largest.

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Problem 7 Solution

7. Sketch the graph of the function $f(x) = \dfrac{x^2}{x^2+3}$. Your graph must clearly indicate all local maxima, local minima, inflection points, and asymptotes. Include the x- and y-coordinates of the local extrema and inflection points.

Solution: The critical points of $f(x)$ are the values of x for which either $f'(x)$ does not exist or $f'(x) = 0$.

$$\begin{aligned}
f'(x) &= 0 \\
\left(\frac{x^2}{x^2+3}\right)' &= 0 \\
\frac{(x^2+3)(x^2)' - (x^2)(x^2+3)'}{(x^2+3)^2} &= 0 \\
\frac{(x^2+3)(2x) - (x^2)(2x)}{(x^2+3)^2} &= 0 \\
\frac{6x}{(x^2+3)^2} &= 0 \\
6x &= 0 \\
x &= 0
\end{aligned}$$

Thus, $\boxed{x = 0}$ is the only critical point of f. (Note: $x^2 + 3 > 0$ for all x.)

The domain of f is $(-\infty, \infty)$. We now split the domain into the intervals $(-\infty, 0)$ and $(0, \infty)$. We then evaluate $f'(x)$ at a test point in each interval to determine the intervals of monotonicity.

Interval	**Test Point**, c	$f'(c)$	**Sign of** $f'(c)$
$(-\infty, 0)$	-1	$f'(-1) = -\frac{3}{8}$	$-$
$(0, \infty)$	1	$f'(1) = \frac{3}{8}$	$+$

Using the table, we conclude that f is increasing on $(0, \infty)$ because $f'(x) > 0$ for all $x \in (0, \infty)$ and f is decreasing on $(-\infty, 0)$ because $f'(x) < 0$ for all $x \in (-\infty, 0)$. Also, since f' changes sign from $-$ to $+$ at $x = 0$ the First Derivative Test implies that $f(0) = 0$ is a local minimum.

To determine the intervals of concavity we start by finding solutions to the equation $f''(x) = 0$

and where $f''(x)$ does not exist.

$$\begin{aligned}
f''(x) &= 0 \\
\left(\frac{6x}{(x^2+3)^2}\right)' &= 0 \\
\frac{(x^2+3)^2(6x)' - (6x)[(x^2+3)^2]'}{(x^2+3)^4} &= 0 \\
\frac{(x^2+3)^2(6) - (6x)(2(x^2+3)(2x))}{(x^2+3)^4} &= 0 \\
\frac{6(x^2+3) - 24x^2}{(x^2+3)^3} &= 0 \\
6(x^2+3) - 24x^2 &= 0 \\
x^2 + 3 - 4x^2 &= 0 \\
-3x^2 + 3 &= 0 \\
x^2 &= 1 \\
x &= \pm 1
\end{aligned}$$

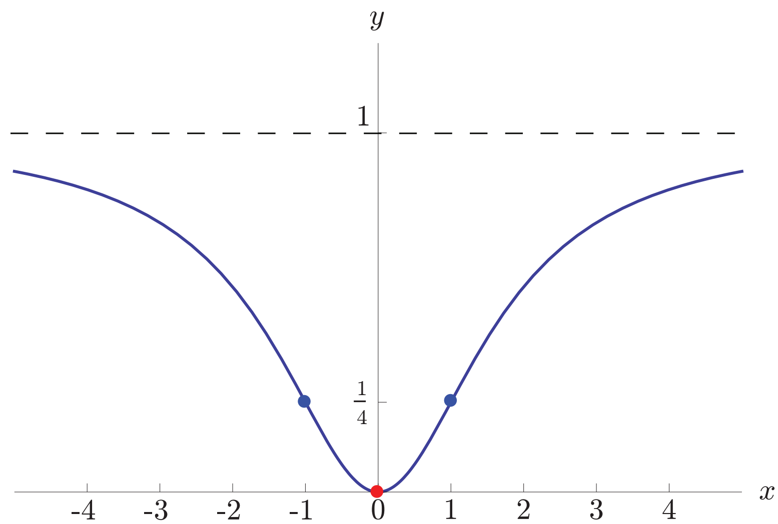
We now split the domain into the intervals $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. We then evaluate $f''(x)$ at a test point in each interval to determine the intervals of concavity.

Interval	**Test Point**, c	$f''(c)$	**Sign of** $f''(c)$
$(-\infty, -1)$	-2	$f''(-2) = -\frac{54}{343}$	$-$
$(-1, 1)$	0	$f''(0) = \frac{2}{3}$	$+$
$(1, \infty)$	2	$f''(2) = -\frac{54}{343}$	$-$

Using the table, we conclude that f is concave down on $(-\infty, -1) \cup (1, \infty)$ because $f''(x) < 0$ for all $x \in (-\infty, -1) \cup (1, \infty)$ and f is concave up on $(-1, 1)$ because $f''(x) > 0$ for all $x \in (-1, 1)$. The inflection points of $f(x)$ are the points where $f''(x)$ changes sign. We can see in the above table that $f''(x)$ changes sign at $x = \pm 1$. Therefore, $\boxed{x = \pm 1}$ are inflection points. The values of $f(x)$ at the inflection points are $f(-1) = f(1) = \frac{1}{4}$.

$f(x)$ has no vertical asymptotes. However, its horizontal asymptote is $y = 1$ because

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2}{x^2+3} = 1$$


y
1
1/4
x
-4
-3
-2
-1
0
1
2
3
4

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Problem 8 Solution

8. A sheep rancher plans to fence a rectangular pasture next to an irrigation canal. No fence will be needed along the canal, but the other three sides must be fenced. The pasture must have an area of 180,000 square yards to provide enough grass for the sheep. Find the dimensions of the pasture which require the least amount of fence.

Solution: We begin by letting x be the length of the side opposite the canal and y be the lengths of the remaining two sides. The function we seek to minimize is the length of the fence:

$$\textbf{Function}: \qquad \text{Length} = x + 2y \tag{1}$$

The constraint in this problem is that the area of the pasture is 180,000 square meters.

$$\textbf{Constraint}: \qquad xy = 180,000 \tag{2}$$

Solving the constraint equation (2) for y we get:

$$y = \frac{180,000}{x} \tag{3}$$

Plugging this into the function (1) and simplifying we get:

$$\begin{aligned} \text{Length} &= x + 2\left(\frac{180,000}{x}\right) \\ f(x) &= x + \frac{360,000}{x} \end{aligned}$$

We want to find the absolute minimum of $f(x)$ on the **interval** $(0, \infty)$. We choose this interval because x must be nonnegative (x represents a length) and non-zero (if x were 0, then the area would be 0 but it must be $180,000$).

The absolute minimum of $f(x)$ will occur either at a critical point of $f(x)$ in $(0, \infty)$ or it will not exist because the interval is open. The critical points of $f(x)$ are solutions to $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 0 \\ \left(x + \frac{360,000}{x}\right)' &= 0 \\ 1 - \frac{360,000}{x^2} &= 0 \\ x^2 &= 360,000 \\ x &= \pm 600 \end{aligned}$$

However, since $x = -600$ is outside $(0, \infty)$, the only critical point is $x = 600$. Plugging this into $f(x)$ we get:

$$f(600) = 600 + \frac{360,000}{600} = 1200$$

Taking the limits of $f(x)$ as x approaches the endpoints we get:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x + \frac{360,000}{x}\right) = 0 + \infty = \infty$$
$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty}\left(x + \frac{360,000}{x}\right) = \infty + 0 = \infty$$

both of which are larger than 1200. We conclude that the length is an absolute minimum at $\boxed{x = 600}$ and that the resulting length is 1200. The last step is to find the corresponding value for y by plugging $x = 600$ into equation (3).

$$y = \frac{180,000}{x} = \frac{180,000}{600} = \boxed{300}$$

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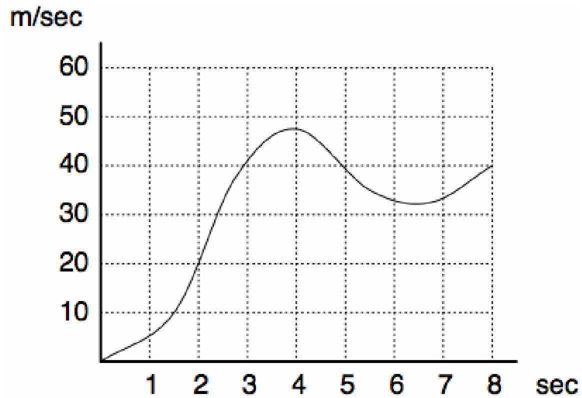
Problem 9 Solution

9. The graph below shows the velocity of a race car, in meters in second, during a time interval of 8 seconds. Estimate the distance that the car traveled during this time interval. You must explain how you arrived at your estimate to receive credit.



Solution: We will use L_8 to estimate the distance traveled. Since $N = 8$ and the interval is $[0, 8]$, we know that $\Delta x = 1$. The value of L_8 is then:

$$L_8 = \Delta x\,[f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7)]$$
$$L_8 = 1 \cdot [0 + 5 + 20 + 20 + 47 + 40 + 33 + 34]$$
$$L_8 = 199 \text{ meters}$$