Math 181, Exam 1, Fall 2013
Problem 1 Solution

1. Compute the integrals

(a) $\int \sin^{-1}(x) \, dx$

(b) $\int \frac{dx}{x^2(x + 1)}$

(c) $\int_0^3 \sqrt{9 - x^2} \, dx$

Solution:

(a) Use Integration by Parts to evaluate the integral. Letting $u = \sin^{-1}(x)$ and $dv = dx$ yields

$$du = \frac{1}{\sqrt{1 - x^2}} \, dx, \quad v = x.$$ 

Then we have

$$\int u \, dv = uv - \int v \, du$$

$$\int \sin^{-1}(x) \, dx = x \sin^{-1}(x) - \int \frac{x}{\sqrt{1 - x^2}} \, dx.$$ 

To evaluate the integral on the right hand side of the above equation, we let $u = 1 - x^2$ and $du = -2x \, dx$ so $-\frac{1}{2} \, du = x \, dx$. Making these substitutions we obtain:

$$\int \sin^{-1}(x) \, dx = x \sin^{-1}(x) + \int \frac{1}{2\sqrt{u}} \, du$$

$$\int \sin^{-1}(x) \, dx = x \sin^{-1}(x) + \sqrt{u} + C$$

$$\int \sin^{-1}(x) \, dx = x \sin^{-1}(x) + \sqrt{1 - x^2} + C$$

(b) Use the method of partial fractions. The decomposition of the integrand is

$$\frac{1}{x^2(x + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x + 1}.$$ 

After clearing denominators we obtain

$$1 = Ax(x + 1) + B(x + 1) + Cx^2.$$
Letting \( x = 0 \) yields \( B = 1 \) and letting \( x = -1 \) yields \( C = 1 \). After expanding the right hand side of the above equation we obtain

\[ 1 = x^2(A + C) + x(A + B) + B. \]

Equating the coefficient of \( x^2 \) on both sides of the equation yields 0 = \( A + C \) so \( A = -C = -1 \). Thus, the decomposition is

\[ \frac{1}{x^2(x+1)} = -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}. \]

The integral is then

\[
\int \frac{dx}{x^2(x+1)} = \int \left( -\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx
\]

(c) The integral represents one-fourth of the area of a circle of radius 3. That is,

\[
\int_0^3 \sqrt{9-x^2} \, dx = \frac{1}{4} \pi (3)^2 = \frac{9\pi}{4}.
\]

The other method of solution is to use the trigonometric substitution

\[ x = 3 \sin \theta, \quad dx = 3 \cos \theta \, d\theta. \]

When \( x = 0 \) we have \( \sin \theta = 0 \) and, thus, \( \theta = 0 \). When \( x = 3 \) we have \( \sin \theta = 1 \) and, thus, \( \theta = \frac{\pi}{2} \). The definite integral is then converted and evaluated as follows:

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} \sqrt{9-(3 \sin \theta)^2} \cdot 3 \cos \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} \sqrt{9 - 9 \sin^2 \theta} \cdot 3 \cos \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} \sqrt{9(1 - \sin^2 \theta)} \cdot 3 \cos \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} \sqrt{9 \cos^2 \theta} \cdot 3 \cos \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} 3 \cos \theta \cdot 3 \cos \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \int_0^{\pi/2} 9 \cos^2 \theta \, d\theta
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = 9 \left[ \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right]_0^{\pi/2}
\]

\[
\int_0^3 \sqrt{9-x^2} \, dx = \frac{9\pi}{4}
\]
2. Compute the length of the graph of \( f(x) = \frac{e^x + e^{-x}}{2} \) from \( x = 0 \) to \( x = \ln(2) \).

**Solution:** The arclength formula is

\[
L = \int_a^b \sqrt{1 + f'(x)^2} \, dx
\]

where

\[
f'(x) = \frac{e^x - e^{-x}}{2}.
\]

The quantity \( 1 + f'(x)^2 \) simplifies as follows:

\[
1 + f'(x)^2 = 1 + \left( \frac{e^x - e^{-x}}{2} \right)^2
\]

\[
1 + f'(x)^2 = 1 + \frac{(e^x - e^{-x})^2}{4}
\]

\[
1 + f'(x)^2 = 1 + \frac{e^{2x} - 2 + e^{-2x}}{4}
\]

\[
1 + f'(x)^2 = 4 + e^{2x} - 2 + e^{-2x}
\]

\[
1 + f'(x)^2 = \frac{e^{2x} + 2 + e^{-2x}}{4}
\]

\[
1 + f'(x)^2 = \frac{\left( e^x + e^{-x} \right)^2}{4}
\]

Therefore, the arclength is

\[
L = \int_0^{\ln(2)} \sqrt{\left( \frac{e^x + e^{-x}}{2} \right)^2} \, dx
\]

\[
L = \int_0^{\ln(2)} \frac{e^x + e^{-x}}{2} \, dx
\]

\[
L = \left. \frac{e^x - e^{-x}}{2} \right|_0^{\ln(2)}
\]

\[
L = \frac{e^{\ln(2)} - e^{-\ln(2)}}{2}
\]

\[
L = \frac{2 - \frac{1}{2}}{2}
\]

\[
L = \frac{3}{4}
\]
3. Consider the region enclosed by \( y = 5 - x^2 \) the \( y \)-axis and \( y = 1 \). Find the volume of revolution of the resulting solid, when the region is rotated about:

(a) the \( x \)-axis,

(b) the axis \( x = -2 \).

**Solution:**

(a) The volume is obtained using the Washer Method. The corresponding formula is

\[
V = \pi \int_a^b \pi [f(x)^2 - g(x)^2] \, dx.
\]

A sketch of the region enclosed by the given curves is shown below.

From the sketch of the region, we know that \( f(x) = 5 - x^2 \) and \( g(x) = 1 \). Thus, the volume is

\[
V = \pi \int_1^2 \left( (5 - x^2)^2 - 1^2 \right) \, dx
\]

\[
V = \pi \int_1^2 \left( 25 - 10x^2 + x^4 - 1 \right) \, dx
\]

\[
V = \pi \int_1^2 \left( x^4 - 10x^2 + 24 \right) \, dx
\]

\[
V = \pi \left[ \frac{1}{5}x^5 - \frac{10}{3}x^3 + 24x \right]_0^2
\]

\[
V = \pi \left[ \frac{32}{5} - \frac{80}{3} + 48 \right]
\]

\[
V = \frac{416\pi}{15}
\]
(b) Upon rotating about the axis $x = -2$, we use the Shell Method to find the corresponding volume. The formula we use is

$$V = 2\pi \int_{a}^{b} (x + 2)[f(x) - g(x)] \, dx$$

where the shell radius is $x + 2$. Using the definitions of $f(x)$ and $g(x)$ from part (a) we have

$$V = 2\pi \int_{0}^{2} (x + 2)(5 - x^2 - 1) \, dx$$

$$V = 2\pi \int_{0}^{2} (x + 2)(4 - x^2) \, dx$$

$$V = 2\pi \int_{0}^{2} (4x - x^3 + 8 - 2x^2) \, dx$$

$$V = 2\pi \left[ 2x^2 - \frac{1}{4}x^4 + 8x - \frac{2}{3}x^3 \right]_{0}^{2}$$

$$V = 2\pi \left[ 2(4) - \frac{1}{4}(4) + 8(4) - \frac{2}{3}(2)^3 \right]$$

$$V = \frac{88\pi}{3}$$
4. Compute the area of each region below.

(a) the region between \( y = x\sqrt{4-x} \) and the \( x \)-axis from \( x = 0 \) to \( x = 3 \)

(b) the region between the graphs of \( y = 5 - x^2 \) and \( y = 3 - x \)

Solution:

(a) The area of the region is

\[
A = \int_{0}^{3} x\sqrt{4-x} \, dx.
\]

Using the substitution \( u = 4 - x \) we obtain \(-du = dx\) and \( x = 4 - u \). The limits of integration become:

- \( x = 0 \Rightarrow u = 4 - 0 = 4 \)
- \( x = 3 \Rightarrow u = 4 - 3 = 1 \)

Thus, the area is

\[
A = -\int_{4}^{1} (4-u)\sqrt{u} \, du
\]

\[
A = \int_{1}^{4} (4u^{1/2} - u^{3/2}) \, du
\]

\[
A = \left[ \frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2} \right]_{1}^{4}
\]

\[
A = \left[ \frac{8}{3}(4)^{3/2} - \frac{2}{5}(4)^{5/2} \right] - \left[ \frac{8}{3}(1)^{3/2} - \frac{2}{5}(1)^{5/2} \right]
\]

\[
A = \frac{64}{3} - \frac{64}{5} - \frac{8}{3} + \frac{2}{5}
\]

\[
A = \frac{94}{15}
\]

(b) The graphs intersect when \( y = y \). That is,

\[
3 - x = 5 - x^2
\]

\[
x^2 - x - 2 = 0
\]

\[
(x - 2)(x + 1) = 0
\]

\[
x = 2, \, x = -1
\]
The graph of \( y = 5 - x^2 \) is above the graph of \( y = 3 - x \) on the interval \(-1 \leq x \leq 2\). Therefore, the area is

\[
A = \int_{-1}^{2} [(5 - x^2) - (3 - x)] \, dx
\]

\[
A = \int_{-1}^{2} (2 + x - x^2) \, dx
\]

\[
A = \left[ 2x + \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-1}^{2}
\]

\[
A = \left[ 2(2) + \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3 \right] - \left[ 2(-1) + \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3 \right]
\]

\[
A = 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}
\]

\[
A = \frac{9}{2}
\]
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Problem 5 Solution

5. Evaluate the indefinite integral

\[ \int \frac{dx}{e^{2x} + e^x}. \]

Consider using the substitution \( u = e^x \).

Solution: Letting \( u = e^x \) yields \( du = e^x \, dx \). In other words, \( \frac{du}{u} = dx \) since \( u = e^x \). The integral converts as follows:

\[
\int \frac{dx}{e^{2x} + e^x} = \int \frac{du}{u^2 + u} = \int \frac{du}{u(u^2 + u)} = \int \frac{du}{u^2(u + 1)}
\]

This integral was solved in Problem 1(b). The answer is

\[
\int \frac{du}{u^2(u + 1)} = -\ln|u| - \frac{1}{u} + \ln|u + 1| + C
\]

Using the fact that \( u = e^x \) yields

\[
\int \frac{dx}{e^{2x} + e^x} = -\ln|e^x| - \frac{1}{e^x} + \ln|e^x + 1| + C
\]