Math 181, Exam 1, Spring 2009
Problem 1 Solution

1.

(a) Differentiate the function:

$$F(x) = \int_{\sqrt{x}}^{x^2} e^{t^3}\,dt$$

(b) Compute the definite integral:

$$\int_1^5 \left(\frac{17}{x} + 3\right) dx$$

Solution:

(a) Using the Fundamental Theorem of Calculus Part II and the Chain Rule, the derivative of $F(x) = \displaystyle\int_{g(x)}^{h(x)} f(t)\,dt$ is:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt \\ &= f(h(x)) \cdot \frac{d}{dx}h(x) - f(g(x)) \cdot \frac{d}{dx}g(x) \end{aligned}$$

Applying the formula to the given function $F(x)$ we get:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_{\sqrt{x}}^{x^2} e^{t^3}\,dt \\ &= e^{(x^2)^3} \cdot \frac{d}{dx}\left(x^2\right) - e^{(\sqrt{x})^3} \cdot \frac{d}{dx}\left(\sqrt{x}\right) \\ &= \boxed{e^{x^6} \cdot (2x) - e^{x^{3/2}} \cdot \left(\frac{1}{2\sqrt{x}}\right)} \end{aligned}$$

(b) Using the Fundamental Theorem of Calculus Part I, the value of the integral is:

$$\begin{aligned} \int_1^5 \left(\frac{17}{x} + 3\right) dx &= \Big[17\ln|x| + 3x\Big]_1^5 \\ &= [17\ln|5| + 3(5)] - [17\ln|1| + 3(1)] \\ &= 17\ln 5 + 15 - 0 - 3 \\ &= \boxed{17\ln 5 + 12} \end{aligned}$$

Math 181, Exam 1, Spring 2009
Problem 2 Solution

2. Compute the indefinite integrals:

$$\int x\sqrt{1-x}\,dx \qquad \int \sin(5\pi x)\,dx$$

Solution: The first integral is computed using the u-substitution method. Let $u = 1 - x$. Then $du = -\,dx \quad \Rightarrow \quad -\,du = dx$ and $x = 1 - u$. Substituting these into the integral and evaluating we get:

$$\begin{aligned}
\int x\sqrt{1-x}\,dx &= \int (1-u)\sqrt{u}\,(-\,du) \\
&= \int \left(u^{3/2} - u^{1/2}\right)\,du \\
&= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\
&= \boxed{\frac{2}{5}(x-1)^{5/2} - \frac{2}{3}(x-1)^{3/2} + C}
\end{aligned}$$

The second integral is computing using the u-substitution method. Let $u = 5\pi x$. Then $du = 5\pi\,dx \quad \Rightarrow \quad \dfrac{1}{5\pi}\,du = dx$ and we get:

$$\begin{aligned}
\int \sin(5\pi x)\,dx &= \int \sin u \left(\frac{1}{5\pi}\,du\right) \\
&= \frac{1}{5\pi}\int \sin u\,du \\
&= -\frac{1}{5\pi}\cos u + C \\
&= \boxed{-\frac{1}{5\pi}\cos(5\pi x) + C}
\end{aligned}$$

Math 181, Exam 1, Spring 2009
Problem 3 Solution

3. Compute the indefinite integrals:

$$\int \cos^3 x\,dx \qquad \int \sqrt{2x+1}\,dx$$

Solution: The first integral is computed by rewriting the integral using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}\int \cos^3 x\,dx &= \int \cos^2 x \cos x\,dx \\ &= \int (1-\sin^2 x)\cos x\,dx\end{aligned}$$

Now let $u = \sin x$. Then $du = \cos x\,dx$ and we get:

$$\begin{aligned}\int \cos^3 x\,dx &= \int (1-\sin^2 x)\cos x\,dx \\ &= \int (1-u^2)\,du \\ &= u - \frac{1}{3}u^3 + C \\ &= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}\end{aligned}$$

The second integral is computing using the u-substitution method. Let $u = 2x+1$. Then $du = 2\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = dx$ and we get:

$$\begin{aligned}\int \sqrt{2x+1}\,dx &= \int \sqrt{u}\left(\frac{1}{2}\,du\right) \\ &= \frac{1}{2}\int u^{1/2}\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2}\right) + C \\ &= \boxed{\frac{1}{3}(2x+1)^{3/2} + C}\end{aligned}$$

Math 181, Exam 1, Spring 2009
Problem 4 Solution

4. Consider the function $f(x) = x - x^2$ on the interval $[0, 2]$. Compute T_2, M_2, and S_4.

Solution: For the estimates T_2 and M_2, the length of each subinterval of $[0, 2]$ is

$$\Delta x = \frac{b-a}{N} = \frac{2-0}{2} = 1$$

The estimate T_2 is:

$$\begin{aligned} T_2 &= \frac{\Delta x}{2}\left[f(0) + 2f(1) + f(2)\right] \\ &= \frac{1}{2}\left[(0-0^2) + 2(1-1^2) + (2-2^2)\right] \\ &= \boxed{-1} \end{aligned}$$

The estimate M_2 is:

$$\begin{aligned} M_2 &= \Delta x\left[f\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\right] \\ &= 1 \cdot \left[\left(\frac{1}{2} - \left(\frac{1}{2}\right)^2\right) + \left(\frac{3}{2} - \left(\frac{3}{2}\right)^2\right)\right] \\ &= \frac{1}{4} - \frac{3}{4} \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

We can use the following formula to find S_4:

$$S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2$$

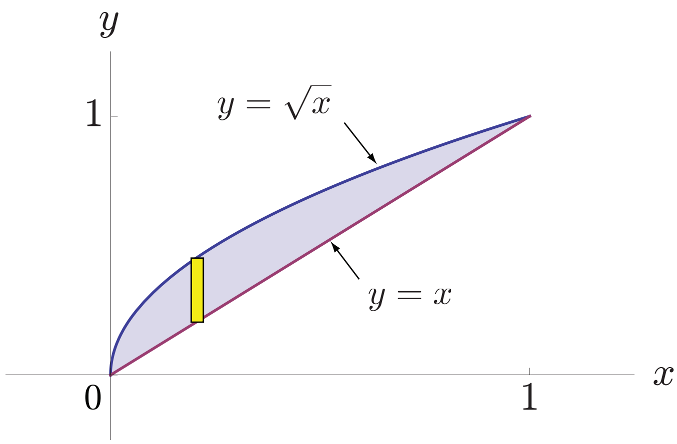
where M_2 and T_2 were found in parts (a) and (b). We get:

$$\begin{aligned} S_4 &= \frac{2}{3}\left(-\frac{1}{2}\right) + \frac{1}{3}(-1) \\ &= \boxed{-\frac{2}{3}} \end{aligned}$$

Math 181, Exam 1, Spring 2009
Problem 5 Solution

5. The region enclosed by the graphs of the functions $y = x$ and $y = \sqrt{x}$ from $x = 0$ to $x = 1$ is rotated about the y-axis. Compute the volume of the resulting solid.

Solution:



We will use the **Shell Method** to compute the volume. The variable of integration is x and the corresponding formula is:

$$V = 2\pi \int_a^b x\,(\text{top} - \text{bottom})\; dx$$

where the top curve is $y = \sqrt{x}$, the bottom curve is $y = x$, and the interval is $0 \leq x \leq 1$. Therefore, the volume is:

$$\begin{aligned} V &= 2\pi \int_0^1 x\left(\sqrt{x} - x\right)\, dx \\ &= 2\pi \int_0^1 \left(x^{3/2} - x^2\right)\, dx \\ &= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^1 \\ &= 2\pi \left[\frac{2}{5} - \frac{1}{3}\right] \\ &= \boxed{\frac{2\pi}{15}} \end{aligned}$$