Math 181, Exam 1, Spring 2010
Problem 1 Solution

1. Evaluate the integral $\int \cos^3 x \, dx$.

Solution: The integral can be solved by rewriting it using the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}\int \cos^3 x \, dx &= \int \cos^2 x \cos x \, dx \\ &= \int \left(1 - \sin^2 x\right) \cos x \, dx\end{aligned}$$

Now let $u = \sin x$. Then $du = \cos x \, dx$ and we get:

$$\begin{aligned}\int \cos^3 x \, dx &= \int \left(1 - \sin^2 x\right) \cos x \, dx \\ &= \int \left(1 - u^2\right) du \\ &= u - \frac{1}{3}u^3 + C \\ &= \boxed{\sin x - \frac{1}{3}\sin^3 x + C}\end{aligned}$$

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Problem 2 Solution

2. Evaluate the integral $\int xe^{3x}\,dx$.

Solution: We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = e^{3x}$. Then $u' = 1$ and $v = \frac{1}{3}e^{3x}$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int xe^{3x}\,dx &= \frac{1}{3}xe^{3x} - \int 1 \cdot \frac{1}{3}e^{3x}\,dx \\ &= \frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\,dx \\ &= \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}\end{aligned}$$

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Problem 3 Solution

3. Evaluate the definite integral $\displaystyle\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx$.

Solution: We evaluate the integral using the u-substitution method. Let $u = 1 + x^3$. Then $du = 3x^2\,dx \quad \Rightarrow \quad \frac{1}{3}\,du = x^2\,dx$. The limits of integration becomes $u = 1 + 0^3 = 1$ and $u = 1 + 2^3 = 9$. We get:

$$\begin{aligned}
\int_0^2 \frac{x^2}{\sqrt{1+x^3}}\,dx &= \int_0^2 \frac{1}{\sqrt{1+x^3}} \cdot x^2\,dx \\
&= \int_1^9 \frac{1}{\sqrt{u}} \cdot \frac{1}{3}\,du \\
&= \frac{1}{3}\int_1^9 u^{-1/2}\,du \\
&= \frac{1}{3}\left[2u^{1/2}\right]_1^9 \\
&= \frac{1}{3}\left[2(9)^{1/2}\right] - \frac{1}{3}\left[2(1)^{1/2}\right] \\
&= \boxed{\frac{4}{3}}
\end{aligned}$$

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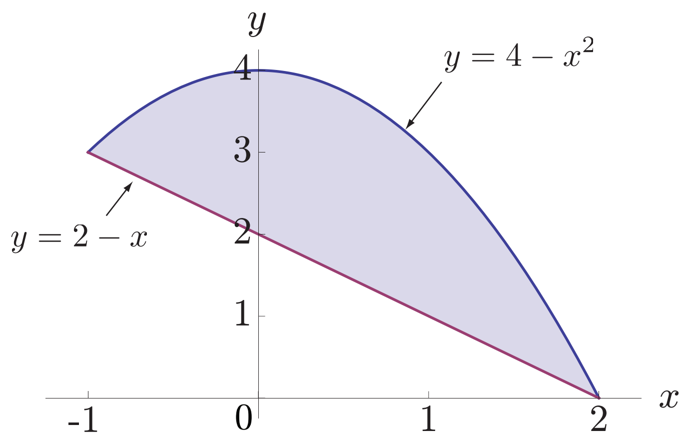
Problem 4 Solution

4. The region R is bounded above by the parabola $y = 4 - x^2$ and bounded below by the line $y = 2 - x$.

(a) Set up a definite integral giving the area of the region R.

(b) Evaluate your integral to find this area.

Solution:

(a) 

The formula we will use to compute the area of the region is:

$$\text{Area} = \int_a^b (\text{top} - \text{bottom})\, dx$$

where the limits of integration are the x-coordinates of the points of intersection of the two curves. These are found by setting the y's equal to each other and solving for x.

$$\begin{aligned} y &= y \\ 2 - x &= 4 - x^2 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x = -1,\ x &= 2 \end{aligned}$$

From the graph we see that the top curve is $y = 4 - x^2$ and the bottom curve is $y = 2 - x$. Therefore, the area is:

$$\begin{aligned} \text{Area} &= \int_a^b (\text{top} - \text{bottom})\, dx \\ &= \int_{-1}^{2} \left[(4 - x^2) - (2 - x)\right] dx \\ &= \int_{-1}^{2} \left(2 + x - x^2\right) dx \end{aligned}$$

(b) The value of the area is:

$$\begin{aligned}
\text{Area} &= \int_{-1}^{2} \left(2 + x - x^2\right)\, dx \\
&= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} \\
&= \left[2(2) + \frac{1}{2}(2)^2 - \frac{1}{3}(2)^3\right] - \left[2(-1) + \frac{1}{2}(-1)^2 - \frac{1}{3}(-1)^3\right] \\
&= \left[4 + 2 - \frac{8}{3}\right] - \left[-2 + \frac{1}{2} + \frac{1}{3}\right] \\
&= \boxed{\frac{9}{2}}
\end{aligned}$$

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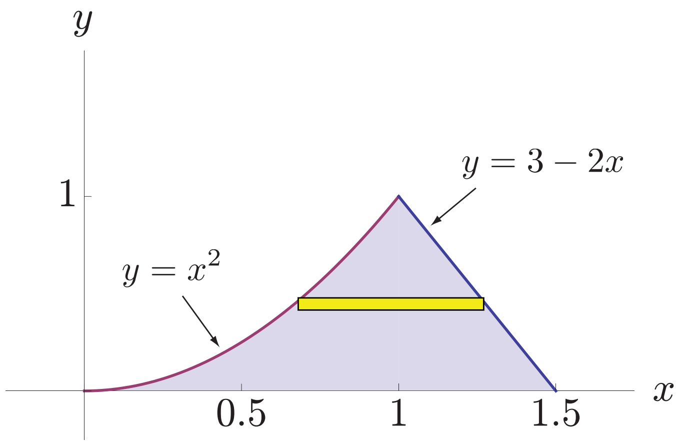
Problem 5 Solution

5. Find the volume of the solid that is obtained by rotating about the x-axis the region enclosed by the graphs of the functions

$$y = 0, \quad y = x^2, \quad \text{and} \quad y = 3 - 2x.$$

Solution: The region being rotated about the x-axis is shown below.



We find the volume using the Shell method. The formula we will use is:

$$V = 2\pi \int_c^d y\,(\text{right} - \text{left})\;dy$$

where the right curve is $y = 3-2x \;\Rightarrow\; x = \frac{1}{2}(3-y)$ and the left curve is $y = x^2 \;\Rightarrow\; x = \sqrt{y}$. The lower limit is $c = 0$. The upper limit is the y-coordinate of the point of intersection in the first quadrant. To find the upper limit, we set the x's equal to each other and solve for y.

$$\begin{aligned} x &= x \\ \sqrt{y} &= \frac{1}{2}(3-y) \\ y &= \frac{1}{4}(3-y)^2 \\ y &= \frac{1}{4}(9-6y+y^2) \\ 4y &= 9 - 6y + y^2 \\ y^2 - 10y + 9 &= 0 \\ (y-9)(y-1) &= 0 \\ y = 9,\; y &= 1 \end{aligned}$$

When $y = 9$ we know that $x = \frac{1}{2}(3 - 9) = -3$. When $y = 1$ we know that $x = \frac{1}{2}(3 - 1) = 1$. Therefore, the upper limit of integration is $d = 1$ since $(1, 1)$ is in the first quadrant.

The volume is then:

$$\begin{aligned}
V &= 2\pi \int_c^d y\,(\text{right} - \text{left})\; dy \\
&= 2\pi \int_0^1 y \left[\frac{1}{2}(3 - y) - \sqrt{y}\right] dy \\
&= 2\pi \int_0^1 \left(\frac{3}{2}y - \frac{1}{2}y^2 - y^{3/2}\right) dy \\
&= 2\pi \left[\frac{3}{4}y^2 - \frac{1}{6}y^3 - \frac{2}{5}y^{5/2}\right]_0^1 \\
&= 2\pi \left[\frac{3}{4}(1)^2 - \frac{1}{6}(1)^3 - \frac{2}{5}(1)^{5/2}\right] \\
&= 2\pi \left[\frac{3}{4} - \frac{1}{6} - \frac{2}{5}\right] \\
&= \boxed{\frac{11\pi}{30}}
\end{aligned}$$

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Problem 6 Solution

6. Compute the average value of the function $f(x) = \sin(\pi x)$ over the interval $[0, 1/2]$.

Solution: The average value is:

$$\begin{aligned}
\bar{f} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{\frac{1}{2}-0}\int_0^{1/2} \sin(\pi x)\,dx \\
&= 2\left[-\frac{1}{\pi}\cos(\pi x)\right]_0^{1/2} \\
&= 2\left[-\frac{1}{\pi}\cos\left(\pi\cdot\frac{1}{2}\right)\right] - 2\left[-\frac{1}{\pi}\cos(\pi\cdot 0)\right] \\
&= 2\left[-\frac{1}{\pi}\cdot 0\right] - 2\left[-\frac{1}{\pi}\cdot 1\right] \\
&= \boxed{\frac{2}{\pi}}
\end{aligned}$$