Math 181, Exam 2, Fall 2011
Problem 1 Solution

1. Compute the arc length of the graph of $f(x) = \sqrt{9 - x^2}$ over $[0, 3]$.

Solution: The arc length can be easily found by recognizing that the graph of the function is a quarter circle of radius 3. Knowing that the arc length of a circle is $2\pi r$, the arc length of $y = f(x)$ is

$$\text{arc length} = \frac{1}{4}2\pi(3) = \boxed{\frac{3\pi}{2}}.$$

One can also resort to finding arc length via the formula

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

where

$$f'(x) = -\frac{x}{\sqrt{9 - x^2}}$$

The arc length is then

$$L = \int_0^3 \sqrt{1 + \left(-\frac{x}{\sqrt{9 - x^2}}\right)^2}\,dx$$
$$L = \int_0^3 \sqrt{1 + \frac{x^2}{9 - x^2}}\,dx$$
$$L = \int_0^3 \sqrt{\frac{9 - x^2 + x^2}{9 - x^2}}\,dx$$
$$L = \int_0^3 \frac{3}{\sqrt{9 - x^2}}\,dx$$

This integral may be solved using the trigonometric substitution $x = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$. Then $\sqrt{9 - x^2} = 3\cos\theta$ and we get

$$L = \int_0^3 \frac{3}{\sqrt{9 - x^2}}\,dx$$
$$L = \int_0^{\pi/2} \frac{3}{3\cos\theta}(3\cos\theta\,d\theta)$$
$$L = \int_0^{\pi/2} 3\,d\theta$$
$$L = \frac{3\pi}{2}$$

Math 181, Exam 2, Fall 2011
Problem 2 Solution

2. Determine the limit of the sequence $a_n = \dfrac{2n^2 + (0.3)^n}{3n^2 - n + 1}$.

Solution: We begin by multiplying the function by $\frac{1}{n^2}$ divided by itself.

$$\frac{2n^2 + (0.3)^n}{3n^2 - n + 1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{2 + \frac{(0.3)^n}{n^2}}{3 - \frac{1}{n} + \frac{1}{n^2}}$$

Using the limit laws for quotients, sums, and differences we find that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2 + \frac{(0.3)^n}{n^2}}{3 - \frac{1}{n} + \frac{1}{n^2}}$$

$$\lim_{n\to\infty} a_n = \frac{\lim\limits_{n\to\infty} 2 + \lim\limits_{n\to\infty} \dfrac{(0.3)^n}{n^2}}{\lim\limits_{n\to\infty} 3 - \lim\limits_{n\to\infty} \dfrac{1}{n} + \lim\limits_{n\to\infty} \dfrac{1}{n^2}}$$

$$\lim_{n\to\infty} a_n = \frac{2+0}{3-0+0}$$

$$\lim_{n\to\infty} a_n = \boxed{\frac{2}{3}}$$

where we note that $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$ for $p > 0$ and $\lim\limits_{n\to\infty} r^n = 0$ for $0 < r < 1$.

Math 181, Exam 2, Fall 2011
Problem 3 Solution

3. Determine whether the improper integral converges, and if so, evaluate it:

(a) $\displaystyle\int_1^{\infty} xe^{-x}\,dx$

(b) $\displaystyle\int_1^{2} \frac{x}{x-1}\,dx$

Solution:

(a) We evaluate the first integral by turning it into a limit calculation.

$$\int_1^{+\infty} xe^{-x}\,dx = \lim_{R\to+\infty} \int_1^{R} xe^{-x}\,dx$$

We use Integration by Parts to compute the integral. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula we get:

$$\begin{aligned}
\int_a^b uv'\,dx &= \Big[uv\Big]_a^b - \int_a^b u'v\,dx \\
\int_1^R xe^{-x}\,dx &= \Big[-xe^{-x}\Big]_1^R - \int_1^R \left(-e^{-x}\right)\,dx \\
&= \Big[-xe^{-x}\Big]_1^R + \int_1^R e^{-x}\,dx \\
&= \Big[-xe^{-x}\Big]_1^R + \Big[-e^{-x}\Big]_1^R \\
&= \Big[-Re^{-R} + 1\cdot e^{-1}\Big] + \Big[-e^{-R} + e^{-1}\Big] \\
&= -\frac{R}{e^R} + \frac{1}{e} - \frac{1}{e^R} + \frac{1}{e} \\
&= -\frac{R}{e^R} - \frac{1}{e^R} + \frac{2}{e}
\end{aligned}$$

We now take the limit of the above function as $R \to +\infty$.

$$\begin{aligned}
\int_1^{+\infty} xe^{-x}\,dx &= \lim_{R\to+\infty} \int_1^R xe^{-x}\,dx \\
&= \lim_{R\to+\infty} \left(-\frac{R}{e^R} - \frac{1}{e^R} + \frac{2}{e}\right) \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} - \lim_{R\to+\infty} \frac{1}{e^R} + \frac{2}{e} \\
&= -\lim_{R\to+\infty} \frac{R}{e^R} - 0 + \frac{2}{e} \\
&\overset{\text{L'H}}{=} -\lim_{R\to+\infty} \frac{(R)'}{(e^R)'} - 0 + \frac{2}{e} \\
&= -\lim_{R\to+\infty} \frac{1}{e^R} - 0 + \frac{2}{e} \\
&= -0 - 0 + \frac{2}{e} \\
&= \boxed{\frac{2}{e}}
\end{aligned}$$

(b) We begin by letting $u = x - 1$. Then $du = dx$ and the limits of integration become $u = 1 - 1 = 0$ and $u = 2 - 1 = 1$. Furthermore, since $u = x - 1$ we have $x = u + 1$. Making these substitutions we get

$$\int_1^2 \frac{x}{x-1}\,dx = \int_0^1 \frac{u+1}{u}\,du = \int_0^1 \left(1 + \frac{1}{u}\right)\,du = \int_0^1 1\,du + \int_0^1 \frac{1}{u}\,du$$

The first integral is proper and evaluates to 1. However, the second integral is improper and diverges because it is a p-integral of the form $\displaystyle\int_0^1 \frac{1}{u^p}\,du$ where $p \geq 1$. Therefore, the given integral **diverges**.

Math 181, Exam 2, Fall 2011
Problem 4 Solution

4. State whether the given series is convergent or not. If convergent find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{2n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{2^n}$

Solution:

(a) We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \sum_{n=1}^{\infty} \frac{1}{4^n} = \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$$

This is a convergent geometric series because $|r| = |\frac{1}{4}| < 1$. We can now use the formula:

$$\sum_{n=M}^{+\infty} cr^n = r^M \cdot \frac{c}{1-r}$$

where $M = 1$, $c = 1$, and $r = \frac{1}{4}$. The sum of the series is then:

$$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \left(\frac{1}{4}\right)^1 \cdot \frac{1}{1-\frac{1}{4}} = \boxed{\frac{1}{3}}$$

(b) We recognize the given series as a geometric series. In order to find its sum we must first rewrite the series.

$$\sum_{n=1}^{\infty} \frac{3^n}{2^n} = \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$$

This is a **divergent** geometric series because $|r| = |\frac{3}{2}| > 1$.

Math 181, Exam 2, Fall 2011
Problem 5 Solution

5. Find the values of x for which the following series converges:

$$\sum_{n=1}^{\infty} \frac{3^n x^n}{n}$$

Solution: We determine the radius of convergence using the Ratio Test.

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3^{n+1}x^{n+1}}{n+1} \cdot \frac{n}{3^n x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{n}{n+1} \cdot \frac{x^{n+1}}{x^n} \right| \\
&= \lim_{n\to\infty} \left| 3 \left(\frac{n}{n+1} \right) x \right| \\
&= \lim_{n\to\infty} \left| 3 \left(\frac{1}{1+\frac{1}{n}} \right) x \right| \\
&= 3|x| \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}} \right) \\
&= 3|x|
\end{aligned}$$

In order to achieve convergence, it must be the case that $\rho = 3|x| < 1$. Therefore, $|x| < \frac{1}{3}$. We must now check the endpoints. Plugging $x = \frac{1}{3}$ into the given power series we get:

$$\sum_{n=1}^{\infty} \frac{3^n (\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which is a divergent p-series ($p = 1 \leq 1$). Plugging in $x = -\frac{1}{3}$ we get:

$$\sum_{n=1}^{\infty} \frac{3^n (-\frac{1}{3})^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

i.e. the alternating harmonic series, which converges by the Leibniz Test. Thus, the interval of convergence is:

$$\boxed{-\frac{1}{3} \leq x < \frac{1}{3}}$$