Math 181, Exam 2, Study Guide
Problem 1 Solution

1. Compute the integrals:

$$\int \frac{1}{x^2-3}\,dx \qquad \int \frac{1}{x^2+3}\,dx \qquad \int \frac{1}{x^2+5x+4}\,dx$$

$$\int \frac{1}{x^2+4x+5}\,dx \qquad \int \frac{x}{x^2+5x+4}\,dx \qquad \int \frac{x}{x^2+4x+5}\,dx$$

$$\int x\sin x\,dx \qquad \int \ln x\,dx \qquad \int x\ln x\,dx$$

$$\int x^2e^{-x}\,dx \qquad \int xe^{-x^2}\,dx \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx$$

Solution:

(a) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2-3} = \frac{1}{(x-\sqrt{3})(x+\sqrt{3})} = \frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}}$$

Next, we multiply the above equation by $(x-\sqrt{3})(x+\sqrt{3})$ to get:

$$1 = A(x+\sqrt{3}) + B(x-\sqrt{3})$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = \sqrt{3}$ and $x = -\sqrt{3}$ for simplicity.

$$x = \sqrt{3}: \;\; A(\sqrt{3}+\sqrt{3}) + B(\sqrt{3}-\sqrt{3}) = 1 \quad \Rightarrow \quad A = \frac{1}{2\sqrt{3}}$$
$$x = -\sqrt{3}: \;\; A(\sqrt{3}-\sqrt{3}) + B(-\sqrt{3}-\sqrt{3}) = 1 \quad \Rightarrow \quad B = -\frac{1}{2\sqrt{3}}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}\int \frac{1}{x^2-3}\,dx &= \int \left(\frac{A}{x-\sqrt{3}} + \frac{B}{x+\sqrt{3}}\right) dx \\ &= \int \left(\frac{\frac{1}{2\sqrt{3}}}{x-\sqrt{3}} + \frac{-\frac{1}{2\sqrt{3}}}{x+\sqrt{3}}\right) dx \\ &= \boxed{\frac{1}{2\sqrt{3}}\ln|x-\sqrt{3}| - \frac{1}{2\sqrt{3}}\ln|x+\sqrt{3}| + C}\end{aligned}$$

(b) We will evaluate this integral using the u-substitution method. Let $u = \dfrac{x}{\sqrt{3}}$. Then $du = \dfrac{1}{\sqrt{3}}\,dx \quad\Rightarrow\quad \sqrt{3}\,du = dx$ and $x = \sqrt{3}\,u$ and we get:

$$\begin{aligned}\int \frac{1}{x^2+3}\,dx &= \int \frac{1}{(\sqrt{3}\,u)^2+3}\left(\sqrt{3}\,du\right) \\ &= \sqrt{3}\int \frac{1}{3u^2+3}\,du \\ &= \frac{\sqrt{3}}{3}\int \frac{1}{u^2+1}\,du \\ &= \frac{\sqrt{3}}{3}\arctan u + C \\ &= \boxed{\frac{\sqrt{3}}{3}\arctan\left(\frac{x}{\sqrt{3}}\right) + C}\end{aligned}$$

(c) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{1}{x^2+5x+4} = \frac{1}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

Next, we multiply the above equation by $(x+1)(x+4)$ to get:

$$1 = A(x+4) + B(x+1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = -1$ and $x = -4$ for simplicity.

$$\begin{aligned}x = -1:&\ \ A(-1+4) + B(-1+1) = 1 \quad\Rightarrow\quad A = \frac{1}{3} \\ x = -4:&\ \ A(-4+4) + B(-4+1) = 1 \quad\Rightarrow\quad B = -\frac{1}{3}\end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}\int \frac{1}{x^2+5x+4}\,dx &= \int \left(\frac{A}{x+1} + \frac{B}{x+4}\right) dx \\ &= \int \left(\frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}}{x+4}\right) dx \\ &= \boxed{\frac{1}{3}\ln|x+1| - \frac{1}{3}\ln|x+4| + C}\end{aligned}$$

(d) We begin by completing the square in the denominator.

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{(x+2)^2+1}\,dx$$

We then evaluate the integral using the u-substitution method. Let $u = x+2$. Then $du = dx$ and we get:

$$\begin{aligned}\int \frac{1}{x^2+4x+5}\,dx &= \int \frac{1}{(x+2)^2+1}\,dx \\ &= \int \frac{1}{u^2+1}\,du \\ &= \arctan u + C \\ &= \boxed{\arctan(x+2) + C}\end{aligned}$$

(e) We will evaluate the integral using Partial Fraction Decomposition. First, we factor the denominator and then decompose the rational function into a sum of simpler rational functions.

$$\frac{x}{x^2+5x+4} = \frac{x}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4}$$

Next, we multiply the above equation by $(x+1)(x+4)$ to get:

$$x = A(x+4) + B(x+1)$$

Then we plug in two different values for x to create a system of two equations in two unknowns (A, B). We select $x = -1$ and $x = -4$ for simplicity.

$$\begin{aligned}x = -1: \;\; A(-1+4) + B(-1+1) = -1 \quad &\Rightarrow \quad A = -\frac{1}{3} \\ x = -4: \;\; A(-4+4) + B(-4+1) = -4 \quad &\Rightarrow \quad B = \frac{4}{3}\end{aligned}$$

Finally, we plug these values for A and B back into the decomposition and integrate.

$$\begin{aligned}
\int \frac{x}{x^2+5x+4}\,dx &= \int \left(\frac{A}{x+1} + \frac{B}{x+4} \right) dx \\
&= \int \left(\frac{-\frac{1}{3}}{x+1} + \frac{\frac{4}{3}}{x+4} \right) dx \\
&= \boxed{-\frac{1}{3}\ln|x+1| + \frac{4}{3}\ln|x+4| + C}
\end{aligned}$$

(f) We begin by completing the square in the denominator.

$$\int \frac{x}{x^2+4x+5}\,dx = \int \frac{x}{(x+2)^2+1}\,dx$$

Now use a little "trick." Add 2 and subtract 2 in the numerator to get:

$$\int \frac{x}{(x+2)^2+1}\,dx = \int \frac{x+2-2}{(x+2)^2+1}\,dx = \int \frac{x+2}{(x+2)^2+1}\,dx - 2\int \frac{1}{(x+2)^2+1}\,dx$$

We then evaluate the integrals using the u-substitution method. Let $u = x+2$. Then $du = dx$ and we get:

$$\begin{aligned}
\int \frac{1}{x^2+4x+5}\,dx &= \int \frac{x+2}{(x+2)^2+1}\,dx - 2\int \frac{1}{(x+2)^2+1}\,dx \\
&= \int \frac{u}{u^2+1}\,du - 2\int \frac{1}{u^2+1}\,du
\end{aligned}$$

To evaluate the first integral we make another substitution. Let $v = u^2+1$. Then $dv = 2u\,du \;\;\Rightarrow\;\; \frac{1}{2}\,dv = u\,du$ and we get:

$$\begin{aligned}
\int \frac{1}{x^2+4x+5}\,dx &= \int \frac{u}{u^2+1}\,du - 2\int \frac{1}{u^2+1}\,du \\
&= \int \frac{1}{v}\left(\frac{1}{2}\,dv\right) - 2\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\int \frac{1}{v}\,dv - 2\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\ln|v| - 2\arctan u + C \\
&= \frac{1}{2}\ln|u^2+1| - 2\arctan u + C \\
&= \frac{1}{2}\ln|(x+2)^2+1| - 2\arctan(x+2) + C \\
&= \boxed{\frac{1}{2}\ln(x^2+4x+5) - 2\arctan(x+2) + C}
\end{aligned}$$

(g) We will evaluate the integral using Integration by Parts. Let $u = x$ and $v' = \sin x$. Then $u' = 1$ and $v = -\cos x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x\sin x\,dx &= -x\cos x - \int(-\cos x)\,dx \\ &= -x\cos x + \int \cos x\,dx \\ &= \boxed{-x\cos x + \sin x + C}\end{aligned}$$

(h) We will evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = 1$. Then $u' = \frac{1}{x}$ and $v = x$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int \frac{1}{x}\cdot x\,dx \\ &= x\ln x - \int dx \\ &= \boxed{x\ln x - x + C}\end{aligned}$$

(i) We will evaluate the integral using Integration by Parts. Let $u = \ln x$ and $v' = x$. Then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x\ln x\,dx &= \frac{1}{2}x^2\ln x - \int \frac{1}{x}\cdot\frac{1}{2}x^2\,dx \\ &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx \\ &= \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}\end{aligned}$$

(j) We evaluate the integral using Integration by Parts. Let $u = x^2$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$. Using the Integration by Parts formula:

$$\int uv'\,dx = uv - \int u'v\,dx$$

we get:

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2 e^{-x} - \int 2x\left(-e^{-x}\right)\,dx \\ &= -x^2 e^{-x} + 2\int x e^{-x}\,dx\end{aligned}$$

A second Integration by Parts must be performed. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$. Using the Integration by Parts formula again we get:

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2 e^{-x} + 2\left[-xe^{-x} - \int \left(-e^{-x}\right)\,dx\right] \\ &= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx \\ &= \boxed{-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C}\end{aligned}$$

(k) We will evaluate this integral using the u-substitution method. Let $u = -x^2$. Then $du = -2x\,dx \quad \Rightarrow \quad -\frac{1}{2}\,du = x\,dx$ and we get:

$$\begin{aligned}\int xe^{-x^2}\,dx &= \int e^u \left(-\frac{1}{2}\,du\right) \\ &= -\frac{1}{2}\int e^u\,du \\ &= -\frac{1}{2}e^u + C \\ &= \boxed{-\frac{1}{2}e^{-x^2} + C}\end{aligned}$$

(l) This one is easy.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

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Problem 2 Solution

2. Use the trapezoid rule and the midpoint rule with one interval ($n = 1$) to estimate:

$$\int \frac{1}{x^2+1}\,dx$$

Solution: Clearly, we're missing some important information – the limits of integration. Let's assume for simplicity that the interval of integration is $[0, 1]$. The length of each subinterval of $[0, 1]$ is:

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{1} = 1$$

The trapezoid approximation T_1 is:

$$\begin{aligned} T_1 &= \frac{\Delta x}{2}\left[f(0) + f(1)\right] \\ &= \frac{1}{2}\left[\frac{1}{0^2+1} + \frac{1}{1^2+1}\right] \\ &= \frac{1}{2}\left[1 + \frac{1}{2}\right] \\ &= \boxed{\frac{3}{4}} \end{aligned}$$

The midpoint approximation M_1 is:

$$\begin{aligned} M_1 &= \Delta x\, f\left(\frac{1}{2}\right) \\ &= (1)\left(\frac{1}{(\frac{1}{2})^2+1}\right) \\ &= (1)\left(\frac{4}{5}\right) \\ &= \boxed{\frac{4}{5}} \end{aligned}$$

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Problem 3 Solution

3. Evaluate the improper integrals:

(a) $\displaystyle\int_0^{\infty} (1+x)^{3/2}\,dx$

(b) $\displaystyle\int_0^{\infty} x^2 e^{-x}\,dx$

Solution:

(a) To evaluate the integral we turn it into a limit calculation.

$$\int_0^{\infty} (1+x)^{3/2}\,dx = \lim_{R\to\infty} \int_0^{R} (1+x)^{3/2}\,dx$$

We use the u-substitution $u = x+1$, $du = dx$ to evaluate the integral. We obtain:

$$\int (1+x)^{3/2}\,dx = \int u^{3/2}\,du = \frac{2}{5}u^{5/2} = \frac{2}{5}(1+x)^{5/2}$$

The definite integral from 0 to R we get:

$$\begin{aligned}\int_0^{R} (1+x)^{3/2}\,dx &= \left[\frac{2}{5}(1+x)^{5/2}\right]_0^R \\ &= \frac{2}{5}(1+R)^{5/2} - \frac{2}{5}(1+0)^{5/2} \\ &= \frac{2}{5}(1+R)^{5/2} - \frac{2}{5}\end{aligned}$$

Taking the limit as $R \to \infty$ we get:

$$\begin{aligned}\int_0^{\infty} (1+x)^{3/2}\,dx &= \lim_{R\to\infty} \int_0^{R} (1+x)^{3/2}\,dx \\ &= \lim_{R\to\infty} \left[\frac{2}{5}(1+R)^{5/2} - \frac{2}{5}\right] \\ &= \infty - \frac{2}{5} \\ &= \infty\end{aligned}$$

Therefore, the integral diverges.

(b) To evaluate the integral we turn it into a limit calculation.

$$\int_0^{\infty} x^2 e^{-x}\,dx = \lim_{R\to\infty} \int_0^{R} x^2 e^{-x}\,dx$$

We use Integration by Parts to evaluate the integral. Let $u = x^2$ and $v' = e^{-x}$. Then $u' = 2x$ and $v = -e^{-x}$ and we get:

$$\begin{aligned}
\int_0^R uv'\,dx &= \Big[uv\Big]_0^R - \int_0^R u'v\,dx \\
\int_0^R x^2e^{-x} &= \Big[-x^2e^{-x}\Big]_0^R - \int_0^R 2x(e^{-x})\,dx \\
&= \Big[-x^2e^{-x}\Big]_0^R + 2\int_0^R xe^{-x}\,dx
\end{aligned}$$

We need another application of Integration by Parts to evaluate the integral on the right hand side above. Let $u = x$ and $v' = e^{-x}$. Then $u' = 1$ and $v = -e^{-x}$ and we get:

$$\begin{aligned}
\int_0^R x^2e^{-x} &= \Big[-x^2e^{-x}\Big]_0^R + 2\int_0^R xe^{-x}\,dx \\
&= \Big[-x^2e^{-x}\Big]_0^R + 2\left\{\Big[-xe^{-x}\Big]_0^R - \int_0^R(-e^{-x})\,dx\right\} \\
&= \Big[-x^2e^{-x}\Big]_0^R + 2\Big[-xe^{-x}\Big]_0^R + 2\int_0^R e^{-x}\,dx \\
&= \Big[-x^2e^{-x}\Big]_0^R + 2\Big[-xe^{-x}\Big]_0^R + 2\Big[-e^{-x}\Big]_0^R \\
&= \Big[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big]_0^R \\
&= \Big[-R^2e^{-R} - 2Re^{-R} - 2e^{-R}\Big] - \Big[-0^2e^{-0} - 2(0)e^{-0} - 2e^{-0}\Big] \\
&= -\frac{R^2}{e^R} - \frac{2R}{e^R} - \frac{2}{e^R} + 2
\end{aligned}$$

Taking the limit as $R \to \infty$ we get:

$$\begin{aligned}
\int_0^\infty x^2 e^{-x}\,dx &= \lim_{R\to\infty} \int_0^R x^2 e^{-x}\,dx \\
&= \lim_{R\to\infty} \left(-\frac{R^2}{e^R} - \frac{2R}{e^R} - \frac{2}{e^R} + 2 \right) \\
&= -\lim_{R\to\infty} \frac{R^2}{e^R} - 2\lim_{R\to\infty} \frac{R}{e^R} - 0 + 2 \\
&\overset{\text{L'H}}{=} -\lim_{R\to\infty} \frac{(R^2)'}{(e^R)'} - 2\lim_{R\to\infty} \frac{(R)'}{(e^R)'} - 0 + 2 \\
&= -\lim_{R\to\infty} \frac{2R}{e^R} - 2\lim_{R\to\infty} \frac{1}{e^R} - 0 + 2 \\
&= -2\lim_{R\to\infty} \frac{R}{e^R} - 2\cdot 0 - 0 + 2 \\
&\overset{\text{L'H}}{=} -2\lim_{R\to\infty} \frac{1}{e^R} - 0 - 0 + 2 \\
&= -2\cdot 0 - 0 - 0 + 2 \\
&= \boxed{2}
\end{aligned}$$

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Problem 4 Solution

4. Compute the arclength of the graph of $y = x^{3/2}$ from $x = 0$ to $x = 5$.

Solution: The arclength is:

$$\begin{aligned} L &= \int_a^b \sqrt{1 + f'(x)^2}\, dx \\ &= \int_0^5 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \\ &= \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx \end{aligned}$$

We now use the u-substitution $u = 1 + \frac{9}{4}x$. Then $\frac{4}{9}\, du = dx$, the lower limit of integration changes from 0 to 1, and the upper limit of integration changes from 5 to $\frac{49}{4}$.

$$\begin{aligned} L &= \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx \\ &= \frac{4}{9}\int_1^{49/4} \sqrt{u}\, du \\ &= \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{49/4} \\ &= \frac{4}{9}\left[\frac{2}{3}\left(\frac{49}{4}\right)^{3/2} - \frac{2}{3}(1)^{3/2}\right] \\ &= \frac{8}{27}\left[\frac{343}{8} - 1\right] \\ &= \boxed{\frac{335}{27}} \end{aligned}$$

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Problem 5 Solution

5. Calculate the area of the surface obtained by rotating the curve $y = x^3$ from $x = 0$ to $x = 4$ about the x-axis.

Solution: The surface area formula is:

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

Using $a = 0$, $b = 4$, $f(x) = x^3$, and $f'(x) = 3x^2$ we get:

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^4 x^3\sqrt{1 + (3x^2)^2}\,dx \\ &= 2\pi \int_0^4 x^3\sqrt{1 + 9x^4}\,dx\end{aligned}$$

To evaluate the integral we use the u-substitution $u = 1 + 9x^4$. Then $\frac{1}{36}\,du = x^3\,dx$, the lower limit of integration changes from 0 to 1, and the upper limit changes from 4 to 2305. Making these substitutions we get:

$$\begin{aligned}\text{Surface Area} &= \frac{2\pi}{36}\int_1^{2305} \sqrt{u}\,du \\ &= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]_1^{2305} \\ &= \frac{\pi}{18}\left[\frac{2}{3}(2305)^{3/2} - \frac{2}{3}(1)^{3/2}\right] \\ &= \boxed{\frac{\pi}{27}\left[(2305)^{3/2} - 1\right]}\end{aligned}$$

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Problem 6 Solution

6. A fish tank is filled to a height of 1 foot with water which weighs 62.5 pounds per cubic foot. One side is a vertical rectangular sheet of glass with 2 square feet below the water level. What is the force of the water on this side? (Set up and evaluate the integral.)

Solution: We put the origin of the coordinate system at the vertex of the triangle at the water surface and define the positive y direction as being downward. The fluid force is then:

$$F = w\int_a^b y f(y)\,dy$$

where $w = 62.5$, $a = 0$, $b = 1$, and $f(y) = 2$ is the length of a horizontal strip of the plate at a depth of y from the water surface. The fluid force is then:

$$F = 62.5\int_0^1 y(2)\,dy$$
$$F = 62.5\int_0^1 2y\,dy$$
$$F = 62.5\Big[y^2\Big]_0^1$$
$$\boxed{F = 62.5 \text{ pounds}}$$