Math 181, Final Exam, Fall 2011
Problem 1 Solution

1. Find the limit of the following sequences as $n \to \infty$.

(a) $a_n = \dfrac{3n^4 - n^3 + 2}{2n^4 + n^2 - 10}$

(b) $b_n = \dfrac{n + \sin(n)}{2n^2 - n + 1}$

Solution:

(a) We proceed by multiplying the function by $\frac{1}{n^4}$ divided by itself and then use the fact that $\lim\limits_{n\to\infty} \dfrac{c}{n^p} = 0$ for any constant c and any positive number p.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n^4 - n^3 + 2}{2n^4 + n^2 - 10} \cdot \frac{\frac{1}{n^4}}{\frac{1}{n^4}},$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3 - \frac{1}{n} + \frac{2}{n^4}}{2 + \frac{1}{n^2} - \frac{10}{n^2}},$$

$$\lim_{n\to\infty} a_n = \frac{3 - 0 + 0}{2 + 0 - 0},$$

$$\boxed{\lim_{n\to\infty} a_n = \frac{3}{2}.}$$

(b) We begin by multiplying the given function by $\frac{1}{n^2}$ divided by itself.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n + \sin(n)}{2n^2 - n + 1} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}},$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{\sin(n)}{n^2}}{2 - \frac{1}{n} + \frac{1}{n^2}}.$$

We know that the limits of $\frac{1}{n}$ and $\frac{1}{n^2}$ as $n \to \infty$ are both 0 using the fact that $\lim\limits_{n\to\infty} \dfrac{c}{n^p} = 0$ for any constant c and any positive number p.

We use the Squeeze Theorem to evaluate the limit of $\frac{\sin(n)}{n^2}$ as $n \to \infty$. To begin, we note that $-1 \le \sin(n) \le 1$ for all n. We then divide each part of the inequality by n^2 to obtain

$$-\frac{1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}.$$

The limits of $-\frac{1}{n^2}$ and $\frac{1}{n^2}$ as $n \to \infty$ are both 0. Thus, the limit of $\frac{\sin(n)}{n^2}$ as $n \to \infty$ is also 0 by the Squeeze Theorem.

The value of the limit is then:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\frac{1}{n} + \frac{\sin(n)}{n^2}}{2 - \frac{1}{n} + \frac{1}{n^2}},$$
$$\lim_{n\to\infty} b_n = \frac{0+0}{2-0+0},$$
$$\boxed{\lim_{n\to\infty} b_n = 0.}$$

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Problem 2 Solution

2. Determine whether each series converges or diverges. Justify your answer.

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{2^k}$

Solution:

(a) We will use the Integral Test to show that the integral diverges. The function $f(x) = \frac{1}{x \ln x}$ is positive and decreasing for $x \geq 2$ and the value of the integral of $f(x)$ on $[2, \infty)$ is:

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \int_2^b \frac{1}{x \ln x}\, dx,$$
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \Big[\ln(\ln x)\Big]_2^b,$$
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{b \to \infty} \Big[\ln(\ln b) - \ln(\ln 2)\Big],$$
$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \infty$$

Thus, since the integral diverges we know that the series diverges by the Integral Test.

Note: The antiderivative of $\frac{1}{x \ln x}$ was determined using the substitution $u = \ln x$, $du = \frac{1}{x}\, dx$.

(b) The series is alternating so we will use the Alternating Series Test to show that it converges. First, we note that $f(k) = \frac{k^2}{2^k}$ is positive and decreasing for $k \geq 1$. Also,

$$\lim_{k \to \infty} \frac{k^2}{2^k} = 0$$

because we know that exponential functions grow much faster than polynomials. Therefore, the series converges by the Alternating Series Test.

Note: An alternative solution is to show that the series converges absolutely by testing the series $\sum \frac{k^2}{2^k}$ using, for example, the Ratio Test.

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Problem 3 Solution

3. Determine whether each improper integral converges or diverges. Justify your answer.

(a) $\displaystyle\int_0^2 \frac{1}{\sqrt{x}(x-2)}\,dx$

(b) $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\,dx$

Solution:

(a) The integrand is undefined at both limits of integration so we begin by splitting the integral into two integrals.

$$\int_0^2 \frac{1}{\sqrt{x}(x-2)}\,dx = \int_0^1 \frac{1}{\sqrt{x}(x-2)}\,dx + \int_1^2 \frac{1}{\sqrt{x}(x-2)}\,dx$$

An antiderivative for $\frac{1}{\sqrt{x}(x-2)}$ is found by letting $u = \sqrt{x}$, $u^2 = x$, and $2\,du = \frac{1}{\sqrt{x}}\,dx$. Making these substitutions we get

$$\int \frac{1}{\sqrt{x}(x-2)}\,dx = \int \frac{2}{u^2-2}\,du$$

We now use the Method of Partial Fractions to evaluate the integral on the right hand side. Omitting the details of the decomposition, we end up with

$$\frac{2}{u^2-2} = \frac{\frac{1}{\sqrt{2}}}{u-\sqrt{2}} - \frac{\frac{1}{\sqrt{2}}}{u+\sqrt{2}}.$$

The antiderivative of $\frac{1}{\sqrt{x}(x-2)}$ is then

$$\begin{aligned}
\int \frac{1}{\sqrt{x}(x-2)}\,dx &= \int \frac{2}{u^2-2}\,du, \\
&= \int \left(\frac{\frac{1}{\sqrt{2}}}{u-\sqrt{2}} - \frac{\frac{1}{\sqrt{2}}}{u+\sqrt{2}} \right) du, \\
&= \frac{1}{\sqrt{2}} \int \left(\frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}} \right) du, \\
&= \frac{1}{\sqrt{2}} \left(\ln|u-\sqrt{2}| - \ln|u+\sqrt{2}| \right), \\
&= \frac{1}{\sqrt{2}} \left(\ln|\sqrt{x}-\sqrt{2}| - \ln|\sqrt{x}+\sqrt{2}| \right), \\
&= \frac{1}{\sqrt{2}} \ln\left| \frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}} \right|.
\end{aligned}$$

Returning to the integral $\int_1^2 \frac{1}{\sqrt{x}(x-2)}\,dx$ we find that

$$\int_1^2 \frac{1}{\sqrt{x}(x-2)}\,dx = \lim_{b\to 2^-} \int_1^b \frac{1}{\sqrt{x}(x-2)}\,dx,$$
$$\int_1^2 \frac{1}{\sqrt{x}(x-2)}\,dx = \lim_{b\to 2^-} \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{b}-\sqrt{2}}{\sqrt{b}+\sqrt{2}}\right| - \frac{1}{\sqrt{2}}\ln\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right|.$$

The limit of the first term is $-\infty$ because the term inside the natural logarithm tends to 0 as $b\to 2^-$. The second term is constant so it will remain constant in the limit as $b\to 2^-$. Therefore, the value of the integral is

$$\int_1^2 \frac{1}{\sqrt{x}(x-2)}\,dx = -\infty$$

Since this integral diverges, the integral $\int_0^2 \frac{1}{\sqrt{x}(x-2)}\,dx$ diverges as well.

(b) We begin by rewriting the integral as a limit.

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\arctan x}{x^2}\,dx$$

We now focus on finding an antiderivative of $\frac{\arctan x}{x^2}$ using Integration by Parts. Letting $u = \arctan x$ and $dv = \frac{1}{x^2}\,dx$ we get $du = \frac{1}{1+x^2}\,dx$ and $v = -\frac{1}{x}$. Using the Integration by Parts formula we find that

$$\int u\,dv = uv - \int v\,du,$$
$$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x} + \int \frac{1}{x(1+x^2)}\,dx.$$

The integral on the right hand side is evaluated using the Method of Partial Fractions.

$$\int \frac{1}{x(1+x^2)}\,dx = \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right)\,dx,$$
$$\int \frac{1}{x(1+x^2)}\,dx = \ln|x| - \frac{1}{2}\ln|x^2+1|,$$
$$\int \frac{1}{x(1+x^2)}\,dx = \ln\left|\frac{x}{\sqrt{x^2+1}}\right|.$$

Thus, an antiderivative of $\frac{\arctan x}{x^2}$ is

$$\int \frac{\arctan x}{x^2}\,dx = -\frac{\arctan x}{x} + \ln\left|\frac{x}{\sqrt{x^2+1}}\right|.$$

We can now evaluate the improper integral.

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\arctan x}{x^2}\,dx,$$

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{\arctan x}{x}+\ln\left|\frac{x}{\sqrt{x^2+1}}\right|\right]_1^b,$$

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \lim_{b\to\infty}\left[\left(-\frac{\arctan b}{b}+\ln\left|\frac{b}{\sqrt{b^2+1}}\right|\right)-\left(-\frac{\arctan 1}{1}+\ln\left|\frac{1}{\sqrt{1^2+1}}\right|\right)\right],$$

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = \underbrace{\lim_{b\to\infty}\left(-\frac{\arctan b}{b}\right)}_{\to -\frac{\pi/2}{\infty}=0}+\ln\left|\underbrace{\lim_{b\to\infty}\frac{b}{\sqrt{b^2+1}}}_{\to 1}\right|+\frac{\pi}{4}-\ln\left(\frac{1}{\sqrt{2}}\right),$$

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx = 0+\ln|1|+\frac{\pi}{4}-\ln\left(\frac{1}{\sqrt{2}}\right),$$

$$\boxed{\int_1^\infty \frac{\arctan x}{x^2}\,dx = \frac{\pi}{4}-\ln\left(\frac{1}{\sqrt{2}}\right).}$$

Since the integral evaluates to a number we say that it converges.

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Problem 4 Solution

4. Evaluate the following integrals:

(a) $\displaystyle\int (\cos x)^{-1} \sin^3 x \, dx$

(b) $\displaystyle\int \frac{1}{x^2 - 4x - 12} \, dx$

Solution:

(a) We begin by rewriting $\sin^3 x$ as $\sin x \sin^2 x = \sin x(1 - \cos^2 x)$. Now let $u = \cos x$ and $-du = \sin x \, dx$. The integral is then transformed and evaluated as follows:

$$\int (\cos x)^{-1} \sin^3 x \, dx = \int \frac{1}{\cos x} \cdot \sin x(1 - \cos^2 x) \, dx,$$
$$\int (\cos x)^{-1} \sin^3 x \, dx = -\int \frac{1}{u} \cdot (1 - u^2) \, du,$$
$$\int (\cos x)^{-1} \sin^3 x \, dx = -\int \left(\frac{1}{u} - u\right) \, du,$$
$$\int (\cos x)^{-1} \sin^3 x \, dx = -\ln|u| + \frac{1}{2}u^2 + C,$$
$$\boxed{\int (\cos x)^{-1} \sin^3 x \, dx = -\ln|\cos x| + \frac{1}{2}\cos^2 x + C.}$$

(b) Using the Method of Partial Fractions we find that

$$\int \frac{1}{x^2 - 4x - 12} \, dx = \int \left(\frac{\frac{1}{8}}{x - 6} - \frac{\frac{1}{8}}{x + 2}\right) \, dx,$$
$$\boxed{\int \frac{1}{x^2 - 4x - 12} \, dx = \frac{1}{8}\ln|x - 6| - \frac{1}{8}\ln|x + 2| + C.}$$

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Problem 5 Solution

5. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by the graphs of $y = 2x - x^2$ and $y = x$.

Solution: We find the volume using the Washer Method.

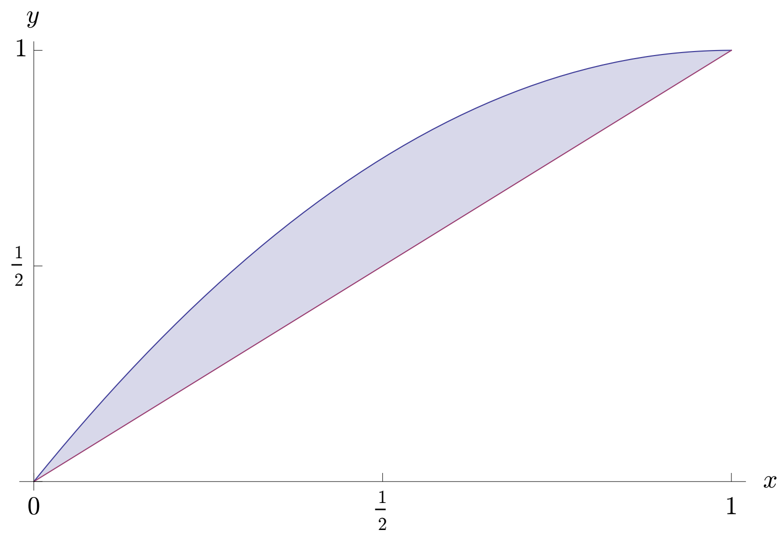
$$V = \int_a^b \pi \left(\text{top}^2 - \text{bottom}^2\right)\, dx$$

From the graph below we see that the bottom curve is $y = x$ and the top curve is $y = 2x - x^2$. The intersection points are determined by setting the two equations equal to one another and solving for x.

$$\begin{aligned} y &= y, \\ x &= 2x - x^2, \\ x^2 - x &= 0, \\ x(x-1) &= 0, \\ x = 0,\ x &= 1. \end{aligned}$$

The volume is then

$$\begin{aligned} V &= \int_0^1 \pi \left(\left(2x - x^2\right)^2 - x^2\right)\, dx, \\ V &= \pi \int_0^1 \left(4x^2 - 4x^3 + x^4 - x^2\right)\, dx, \\ V &= \pi \int_0^1 \left(x^4 - 4x^3 + 3x^2\right)\, dx, \\ V &= \pi \left[\frac{1}{5}x^5 - x^4 + x^3\right]_0^1, \\ V &= \pi \left(\frac{1}{5} - 1 + 1\right), \\ &\boxed{V = \frac{\pi}{5}.} \end{aligned}$$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Problem 6 Solution

6. Find the power series representation centered at 0 for the following functions. Give the interval of convergence of the series.

(a) $f(x) = \dfrac{1}{(1-x)^2}$

(b) $g(x) = x^2 e^{-x}$

Solution:

(a) First we recognize that

$$f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}.$$

Then, using the fact that the power series for $\frac{1}{1-x}$ centered at 0 is

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots,$$

we obtain

$$\begin{aligned}
\frac{1}{(1-x)^2} &= \frac{d}{dx}\frac{1}{1-x}, \\
\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots\right), \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + \cdots, \\
&\boxed{\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} kx^{k-1}.}
\end{aligned}$$

The interval of convergence of the power series for $\frac{1}{1-x}$ is $-1 < x < 1$ so we know that the power series for $\frac{1}{(1-x)^2}$ converges for the same values of x. Upon checking the endpoints $x = -1$ and $x = 1$ we get the two series

$$\sum_{k=1}^{\infty} k(-1)^{k-1} \qquad \text{and} \qquad \sum_{k=1}^{\infty} k$$

which both diverge by the Divergence Test. Thus, the interval of convergence is $-1 < x < 1$.

(b) Using the fact that the power series centered at 0 for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots ,$$

we obtain

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots .$$

Therefore, the power series for $g(x) = x^2 e^{-x}$ is

$$x^2 e^{-x} = x^2 \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right),$$

$$x^2 e^{-x} = x^2 - x^3 + \frac{x^4}{2!} - \frac{x^5}{3!} + \cdots ,$$

$$\boxed{x^2 e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+2}}{k!}.}$$

The interval of convergence of the power series for e^x is $-\infty < x < \infty$. Thus, the interval of convergence of the power series for e^{-x} is also $-\infty < x < \infty$. Multiplication by x^n where n is a positive integer does not change the interval of convergence. Thus, the interval of convergence for $x^2 e^{-x}$ is $-\infty < x < \infty$.

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Problem 7 Solution

7. Let $f(x) = \cos(2x) - 1 + 2x^2$.

(a) Find the first two non-zero terms in the Maclaurin series expansion of f.

(b) Using the expansion found in step (a) compute the limit:

$$\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4}$$

Solution:

(a) Using the fact that the Maclaurin series for $\cos(x)$ is

$$\cos(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

we have

$$\begin{aligned}
\cos(2x) &= \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{(2k)!}, \\
\cos(2x) &= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots, \\
\cos(2x) &= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots.
\end{aligned}$$

Therefore, the first two non-zero terms in the Maclaurin series expansion of f are

$$\begin{aligned}
f(x) &= \cos(2x) - 1 + 2x^2, \\
f(x) &= \left(1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots\right) - 1 + 2x^2, \\
&\boxed{f(x) = \frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots.}
\end{aligned}$$

(b) Using the expansion from part (a), we evaluate the limit as follows:

$$\begin{aligned}
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} &= \lim_{x \to 0} \frac{\frac{2}{3}x^4 - \frac{4}{45}x^6 + \cdots}{x^4}, \\
\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} &= \lim_{x \to 0} \left(\frac{2}{3} - \frac{4}{45}x^2 + \cdots\right), \\
&\boxed{\lim_{x \to 0} \frac{\cos(2x) - 1 + 2x^2}{x^4} = \frac{2}{3}.}
\end{aligned}$$

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Problem 8 Solution

8. An equation of a curve in polar coordinates is given by

$$r = 2\cos\theta, \quad 0 \le \theta \le 2\pi.$$

(a) Rewrite the equation in Cartesian coordinates. Sketch and identify the curve.

(b) Find the arc length of the curve using the integral formula.

Solution:

(a) To rewrite the equation in Cartesian coordinates we begin by multiplying both sides of the equation by r to get

$$r^2 = 2r\cos\theta.$$

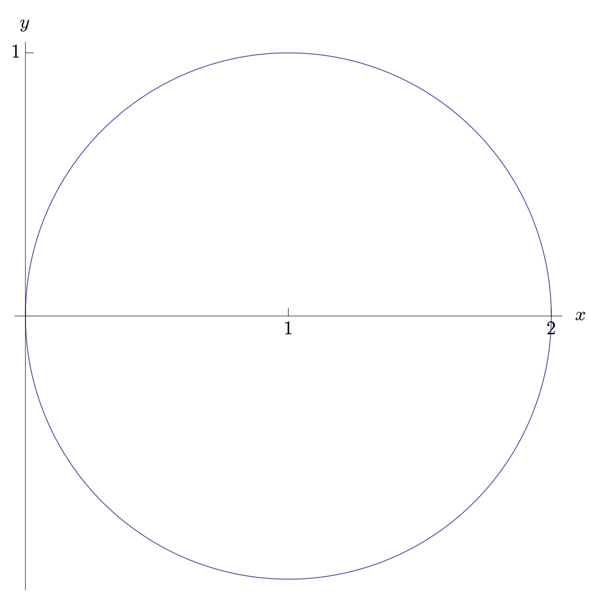
Then, recognizing the fact that $x^2 + y^2 = r^2$ and that $x = r\cos\theta$ we get

$$x^2 + y^2 = 2x$$

To take things a step further, we put the $2x$ to the left hand side and complete the square to get

$$\begin{aligned} x^2 + y^2 &= 2x, \\ x^2 - 2x + y^2 &= 0, \\ (x-1)^2 - 1 + y^2 &= 0, \\ \boxed{(x-1)^2 + y^2 = 1}, \end{aligned}$$

which we recognize is a circle centered at $(1, 0)$ with radius 1. A plot of the curve is sketched below.



(b) The arc length formula for a curve $r = f(\theta)$ defined on the interval $\alpha \leq \theta \leq \beta$ in polar coordinates is

$$L = \int_{\alpha}^{\beta} \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$

In this case, the function is $f(\theta) = 2\cos\theta$ so that $f'(\theta) = -2\sin\theta$. The arc length is then

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta, \\
L &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta}\, d\theta, \\
L &= \int_0^{2\pi} \sqrt{4(\cos^2\theta + \sin^2\theta)}\, d\theta, \\
L &= \int_0^{2\pi} \sqrt{4(1)}\, d\theta, \\
L &= \int_0^{2\pi} 2\, d\theta, \\
L &= 2\theta\Big|_0^{2\pi}, \\
&\boxed{L = 4\pi.}
\end{aligned}$$