Math 181, Final Exam, Spring 2012
Problem 1 Solution

1. Evaluate the following indefinite integral.

$$\int \frac{4x^2+3x-1}{(x-1)^2(x+2)}\,dx$$

Solution: We use the Method of Partial Fractions to evaluate the integral. First, we note that the integrand is a proper rational function. Next, we note that the denominator is already factored. We can now decompose the function into a sum of simpler rational functions as follows:

$$\frac{4x^2+3x-1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

noting that $x-1$ is a repeated linear factor. We begin the process of determining the constants A, B, and C by multiplying the above equation by the denominator on the left hand side.

$$4x^2+3x-1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

Upon letting $x=1$ we find that

$$\begin{aligned}
4(1)^2+3(1)-1 &= A(1-1)(1+2)+B(1+2)+C(1-1)^2,\\
4+3-1 &= 0+3B+0,\\
6 &= 3B,\\
B &= 2.
\end{aligned}$$

Upon letting $x=-2$ we find that

$$\begin{aligned}
4(-2)^2+3(-2)-1 &= A(-2-1)(-2+2)+B(-2+2)+C(-2-1)^2,\\
16-6-1 &= 0+0+9C,\\
9 &= 9C,\\
C &= 1.
\end{aligned}$$

Now that we have the values of B and C, we can determine the value of A by plugging in any value of x we wish. For simplicity, we choose $x=0$:

$$\begin{aligned}
4(0)^2+3(0)-1 &= A(0-1)(0+2)+2(0+2)+1(0-1)^2,\\
-1 &= -2A+4+1,\\
-6 &= -2A,\\
A &= 3.
\end{aligned}$$

We can now evaluate the indefinite integral.

$$\int \frac{4x^2+3x-1}{(x-1)^2(x+2)}\,dx = \int \left(\frac{3}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{x+2}\right)dx,$$

$$\boxed{\int \frac{4x^2+3x-1}{(x-1)^2(x+2)}\,dx = 3\ln|x-1| - \frac{2}{x-1} + \ln|x+2| + C.}$$

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Problem 2 Solution

2. Evaluate the following indefinite integral.

$$\int \frac{dx}{x^2\sqrt{x^2-4}}$$

Solution: We a trigonometric substitution to evaluate the integral. Since the denominator has a term of the form $x^2 - a^2$ where $a = 2$, we let $x = 2\sec\theta$ so that $dx = 2\sec\theta\tan\theta\, d\theta$. Making the proper substitutions into the integral, using the trigonometric identity $\sec^2\theta - 1 = \tan^2\theta$, simplifying, and evaluating we find that

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{(2\sec\theta)^2\sqrt{(2\sec\theta)^2-4}}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta\sqrt{4}\sqrt{\sec^2\theta-1}}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^2\theta\sqrt{\tan^2\theta}}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^2\theta\tan\theta}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \frac{1}{4}\int \frac{d\theta}{\sec\theta}, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \frac{1}{4}\int \cos\theta\, d\theta, \\
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \frac{1}{4}\sin\theta + C.
\end{aligned}$$

Finally, we must write our answer in terms of x using the fact that $x = 2\sec\theta$. From this equation we determine that

$$x = 2\sec\theta \quad \Longleftrightarrow \quad \cos\theta = \frac{2}{x}$$

and using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ we get the following

$$\begin{aligned}
\sin^2\theta &= 1 - \cos^2\theta, \\
\sin^2\theta &= 1 - \frac{4}{x^2}, \\
\sin^2\theta &= \frac{x^2-4}{x^2}, \\
\sin\theta &= \frac{\sqrt{x^2-4}}{x}
\end{aligned}$$

noting that (1) if $x < -2$ then $\theta \in (-\frac{\pi}{2}, 0]$ in which case $\sin\theta < 0$ and (2) if $x > 2$ then $\theta \in [0, \frac{\pi}{2})$ in which case $\sin\theta > 0$, which is consistent with the above equation for $\sin\theta$.

Our final answer is:

$$\boxed{\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{1}{4} \cdot \frac{\sqrt{x^2-4}}{x} + C.}$$

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Problem 3 Solution

3. Consider the function

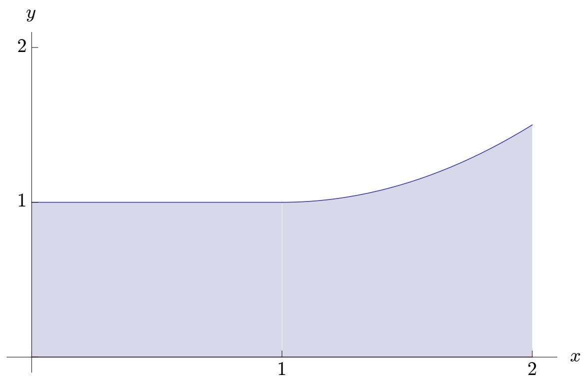
$$f(x) = \begin{cases} 1 & \text{if } x \leq 1 \\ 1 + \frac{1}{2}(x-1)^2 & \text{if } x > 1 \end{cases}.$$

Let R be the region under the graph of $f(x)$ and above the x-axis between $x = 0$ and $x = 2$.

(a) Sketch the graph of $f(x)$ and shade the region R.

(b) Compute the volume of the region obtained by revolving R about the y-axis.

Solution:

(a) 

(b) We compute the volume using the Shell Method. Since we revolve about the y-axis, our variable of integration is x. Furthermore, since the curve that bounds the region from above changes at $x = 1$ we must use two integrals. The set-up and evaluation goes as follows:

$$
\begin{aligned}
V &= 2\pi \int_0^1 x \cdot 1 \, dx + 2\pi \int_1^2 x \cdot \left[1 + \frac{1}{2}(x-1)^2 \right] \, dx, \\
V &= 2\pi \int_0^1 x \, dx + 2\pi \int_1^2 \left(x + \frac{1}{2}x^3 - x^2 + \frac{1}{2}x \right) \, dx, \\
V &= 2\pi \int_0^1 x \, dx + 2\pi \int_1^2 \left(\frac{1}{2}x^3 - x^2 + \frac{3}{2}x \right) \, dx, \\
V &= 2\pi \left[\frac{1}{2}x^2 \right]_0^1 + 2\pi \left[\frac{1}{8}x^4 - \frac{1}{3}x^3 + \frac{3}{4}x^2 \right]_1^2, \\
V &= 2\pi \cdot \frac{1}{2}(1)^2 + 2\pi \left[\left(\frac{1}{8}(2)^4 - \frac{1}{3}(2)^3 + \frac{3}{4}(2)^2 \right) - \left(\frac{1}{8}(1)^4 - \frac{1}{3}(1)^3 + \frac{3}{4}(1)^2 \right) \right], \\
V &= \pi + 2\pi \left(2 - \frac{8}{3} + 3 - \frac{1}{8} + \frac{1}{3} - \frac{3}{4} \right), \\
V &= \pi + 2\pi \left(\frac{4}{6} + \frac{6}{6} - \frac{1}{6} - \frac{3}{6} \right), \\
&\boxed{V = \frac{55\pi}{12}.}
\end{aligned}
$$

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Problem 4 Solution

4. Determine whether each of the following series converges or diverges. Moreover, for those that converge, compute their sum. Indicate the method you are using.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{-2}{n(n+1)}$

(b) $\displaystyle\sum_{n=3}^{\infty} \frac{6}{\pi^n}$

(c) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{3k^3}{2k^3+9k^2+2k}\right)^{k/2}$

Solution:

(a) This series is telescoping. Our method of solution entails finding a formula for S_N, the Nth partial sum, and determining its limit as $N \to \infty$. We begin by using the Method of Partial Fractions to rewrite the nth term as a sum of simpler fractions.

$$\frac{-2}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}.$$

Multiplying both sides of the equation by the denominator on the left hand side gives us

$$-2 = A(n+1) + Bn.$$

Letting $n = 0$ leaves us with $A = -2$ and letting $n = -1$ leaves us with $B = 2$. The series can now be rewritten as

$$\sum_{n=1}^{\infty} \frac{-2}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{-2}{n} + \frac{2}{n+1}\right) = 2\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n}\right).$$

The Nth partial sum of the series is

$$\begin{aligned}
S_N &= 2\sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n}\right), \\
S_N &= 2\left[\left(\frac{1}{2} - 1\right) + \left(\frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{4} - \frac{1}{3}\right) + \cdots \right. \\
&\quad \left. + \left(\frac{1}{N-1} - \frac{1}{N-2}\right) + \left(\frac{1}{N} - \frac{1}{N-1}\right) + \left(\frac{1}{N+1} - \frac{1}{N}\right)\right], \\
S_N &= 2\left[-1 + \frac{1}{N+1}\right].
\end{aligned}$$

By definition, the limit of S_N as $N \to \infty$ is the sum of the series.

$$\sum_{n=1}^{\infty} \frac{-2}{n(n+1)} = \lim_{N\to\infty} S_N,$$

$$\sum_{n=1}^{\infty} \frac{-2}{n(n+1)} = \lim_{N\to\infty} 2\left[-1 + \frac{1}{N+1}\right],$$

$$\boxed{\sum_{n=1}^{\infty} \frac{-2}{n(n+1)} = -2.}$$

Therefore, since the sum exists we know that the series converges.

(b) We begin by rewriting the series as follows:

$$\sum_{n=3}^{\infty} \frac{6}{\pi^n} = \sum_{n=3}^{\infty} 6\left(\frac{1}{\pi}\right)^n,$$

which we recognize is a geometric series with $|r| = |\frac{1}{\pi}| < 1$ meaning that the series converges. However, because the index begins at $n = 3$ we perform a slight maneuver in order to make it begin at 0. Let $n = k + 3$. Then we get

$$\sum_{n=3}^{\infty} 6\left(\frac{1}{\pi}\right)^n = \sum_{k+3=3}^{\infty} 6\left(\frac{1}{\pi}\right)^{k+3} = \sum_{k=0}^{\infty} 6\left(\frac{1}{\pi}\right)^3\left(\frac{1}{\pi}\right)^k.$$

The series is now of the form $\sum_{k=0}^{\infty} ar^k$ whose value is known to be $\frac{a}{1-r}$. Recognizing that $a = 6\left(\frac{1}{\pi}\right)^3$ and $r = \frac{1}{\pi}$ we find the sum of the series to be:

$$\sum_{k=0}^{\infty} 6\left(\frac{1}{\pi}\right)^3\left(\frac{1}{\pi}\right)^k = \frac{6(\frac{1}{\pi})^3}{1-\frac{1}{\pi}} = \boxed{\frac{6}{\pi^3 - \pi^2}.}$$

(c) We will show that the series diverges using the Root Test. We're interested in computing the limit of the kth root of the kth term as $k \to \infty$. This is the value we call r

and its value in this case is:

$$
\begin{aligned}
r &= \lim_{k\to\infty} (a_k)^{1/k}, \\
r &= \lim_{k\to\infty} \left[\left(\frac{3k^3}{2k^3+9k^2+2k}\right)^{k/2}\right]^{1/k}, \\
r &= \lim_{k\to\infty} \left(\frac{3k^3}{2k^3+9k^2+2k}\right)^{1/2}, \\
r &= \sqrt{\lim_{k\to\infty} \frac{3k^3}{2k^3+9k^2+2k}}, \\
r &= \sqrt{\lim_{k\to\infty} \frac{3}{2+\frac{9}{k}+\frac{2}{k^2}}}, \\
r &= \sqrt{\frac{3}{2+0+0}}, \\
r &= \sqrt{\frac{3}{2}}.
\end{aligned}
$$

Noting that $r = \sqrt{\frac{3}{2}} > 1$, we determine that the series **diverges** by the Root Test.

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Problem 5 Solution

5. Determine whether the following series (a) converges absolutely, (b) converges conditionally, or (c) diverges. Justify your answer.

$$\sum_{n=20}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$$

Solution: The series converges conditionally. To show this, we begin by showing that the series does not converge absolutely by considering the series of absolute values:

$$\sum_{n=20}^{\infty} \left| \frac{(-1)^n}{\ln(\ln n)} \right| = \sum_{n=20}^{\infty} \frac{1}{\ln(\ln n)}.$$

We note that the following inequality holds for $n \geq 20$:

$$0 \leq \frac{1}{n} \leq \frac{1}{\ln(\ln n)}.$$

Furthermore, the series $\sum \frac{1}{n}$ diverges because it is a p-series with $p = 1 \leq 1$. Therefore, the series of absolute values diverges by the Comparison Test and, thus, the given alternating series does not converge absolutely.

We now turn our attention to the alternating series and use the Alternating Series Test. Let $a_n = \frac{1}{\ln(\ln n)}$.

(1) a_n is positive for all $n \geq 20$,

(2) a_n is non increasing for all $n \geq 20$, and

(3) $\lim\limits_{n \to \infty} a_n = 0$.

Therefore, the series converges by the Alternating Series Test and, because it does not converge absolutely, it **converges conditionally**.

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Problem 6 Solution

6.

(a) Using the Maclaurin series $\frac{1}{1-x} = 1+x+x^2+x^3+\cdots$ valid for $|x|<1$, find the rational function represented by the power series $\sum_{k=1}^{\infty} kx^{k+1}$.

(b) Compute the value of $\sum_{k=1}^{\infty} \frac{k}{2^{k+1}}$.

Solution:

(a) We begin by writing down the first several terms of the series:

$$\sum_{k=1}^{\infty} kx^{k+1} = x^2 + 2x^3 + 3x^4 + 4x^5 + \cdots .$$

Factoring out an x^2 from each term we get

$$\sum_{k=1}^{\infty} kx^{k+1} = x^2(1 + 2x + 3x^2 + 4x^3 + \cdots).$$

We now recognize that the sum in parentheses is the derivative of the Maclaurin Series for $\frac{1}{1-x}$. That is,

$$\begin{aligned} \frac{d}{dx}\frac{1}{1-x} &= \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \cdots), \\ \frac{d}{dx}\frac{1}{1-x} &= 1 + 2x + 3x^2 + 4x^3 + \cdots . \end{aligned}$$

Therefore, the rational function represented by the given power series is

$$\begin{aligned} \sum_{k=1}^{\infty} kx^{k+1} &= x^2(1 + 2x + 3x^2 + 4x^3 + \cdots), \\ \sum_{k=1}^{\infty} kx^{k+1} &= x^2 \frac{d}{dx}\frac{1}{1-x}, \\ \sum_{k=1}^{\infty} kx^{k+1} &= x^2 \cdot \frac{1}{(1-x)^2}, \end{aligned}$$

$$\boxed{\sum_{k=1}^{\infty} kx^{k+1} = \frac{x^2}{(1-x)^2}.}$$

(b) The value of the given series is found by letting $x = \frac{1}{2}$ in the formula we found in part (a). That is,

$$\sum_{k=1}^{\infty} \frac{k}{2^{k+1}} = \sum_{k=1}^{\infty} k \left(\frac{1}{2}\right)^{k+1},$$

$$\sum_{k=1}^{\infty} \frac{k}{2^{k+1}} = \frac{(\frac{1}{2})^2}{(1-\frac{1}{2})^2},$$

$$\boxed{\sum_{k=1}^{\infty} \frac{k}{2^{k+1}} = 1.}$$

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Problem 7 Solution

7. Find the interval of convergence of the power series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}(x-3)^k$.

Solution: The interval of convergence is found using the Ratio Test.

$$\begin{aligned}
r &= \lim_{k\to\infty} \left| a_{k+1} \cdot \frac{1}{a_k} \right|, \\
r &= \lim_{k\to\infty} \left| \frac{(x-3)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-3)^k} \right|, \\
r &= \lim_{k\to\infty} \left| \frac{k^2}{(k+1)^2} \cdot \frac{(x-3)^{k+1}}{(x-3)^k} \right|, \\
r &= \lim_{k\to\infty} \left| \left(\frac{k}{k+1}\right)^2 \cdot (x-3) \right|, \\
r &= |x-3| \lim_{k\to\infty} \left(\frac{1}{1+\frac{1}{k}}\right)^2, \\
r &= |x-3| \left(\frac{1}{1+0}\right)^2, \\
r &= |x-3|.
\end{aligned}$$

From the Ratio Test we know that the series will converge when $r < 1$. Or, in other words, when

$$\begin{aligned}
|x-3| &< 1, \\
-1 < x-3 &< 1, \\
2 < x &< 4.
\end{aligned}$$

However, when $x = 2$ or $x = 4$ the value of r is 1 which is an inconclusive result according to the Ratio Test. Upon substituting $x = 2$ into the power series we get

$$\sum_{k=1}^{\infty} \frac{1}{k^2}(2-3)^k = \sum k = 1^{\infty} \frac{(-1)^k}{k^2}.$$

The series of absolute values $\sum \frac{1}{k^2}$ is a convergent p-series because $p = 2 > 1$. Thus, the series is absolutely convergent and $x = 2$ is in the interval of convergence. Upon substituting $x = 4$ into the power series we get

$$\sum_{k=1}^{\infty} \frac{1}{k^2}(4-3)^k = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This is a convergent p-series as stated above. Therefore, $x = 4$ is also in the interval of convergence. Our final result is that the interval of convergence is

$$\boxed{2 \le x \le 4.}$$

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Problem 8 Solution

8. Find the third order Taylor polynomial of the function $f(x) = x \cdot \sin x$ centered at $x = \pi$.

Solution: The third order Taylor polynomial of $f(x)$ centered at $x = \pi$ has the form:

$$p_3(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3.$$

The following table lists $f(x)$ and its derivatives evaluated at $x = \pi$.

k	$f^{(k)}(x)$	$f^{(k)}(\pi)$
0	$x\sin(x)$	0
1	$x\cos(x) + \sin(x)$	$-\pi$
2	$-x\sin(x) + 2\cos(x)$	-2
3	$-x\cos(x) - 3\sin(x)$	π

Therefore, the third order Taylor polynomial is

$$p_3(x) = f(\pi) + f'(\pi)(x-\pi) + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3,$$
$$p_3(x) = 0 - \pi(x-\pi) - \frac{2}{2!}(x-\pi)^2 + \frac{\pi}{3!}(x-\pi)^3,$$
$$\boxed{p_3(x) = -\pi(x-\pi) - (x-\pi)^2 + \frac{\pi}{6}(x-\pi)^3.}$$

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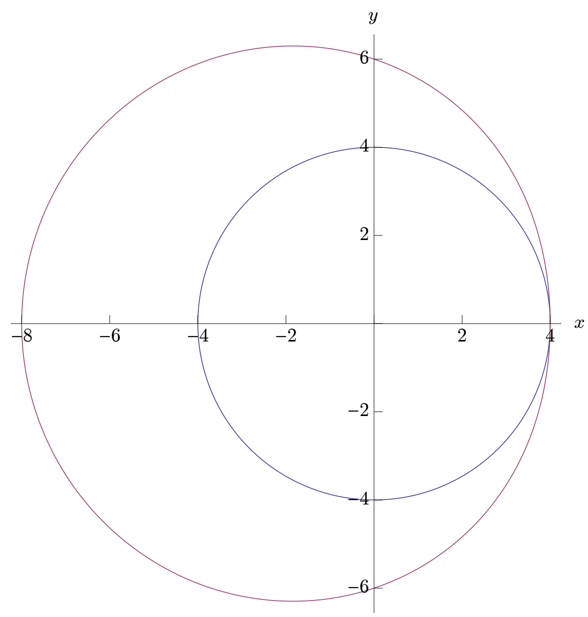
Problem 9 Solution

9. Find the area of the region between the limaçon $r = 6 - 2\cos\theta$ and the circle $r = 4$.

Solution: A plot of the two curves is shown below.



From the plot we see that the limaçon $f(\theta) = 6 - 2\cos\theta$ is the outside function (purple) and the circle $g(\theta) = 4$ is the inside function. Furthermore, the range of θ values is $0 \leq \theta \leq 2\pi$. The area of the region between the curves is then

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left[f(\theta)^2 - g(\theta)^2\right]\,d\theta,$$

$$A = \frac{1}{2}\int_{0}^{2\pi} \left[(6 - 2\cos\theta)^2 - 4^2\right]\,d\theta,$$

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(36 - 24\cos\theta + 4\cos^2\theta - 16\right)\,d\theta,$$

$$A = \frac{1}{2}\int_{0}^{2\pi} \left(20 - 24\cos\theta + 4\cos^2\theta\right)\,d\theta,$$

$$A = \int_{0}^{2\pi} \left(10 - 12\cos\theta + 2\cos^2\theta\right)\,d\theta,$$

$$A = \left[10\theta - 12\sin\theta + 2\left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right)\right]_0^{2\pi},$$

$$A = \left[11\theta - 12\sin\theta + \frac{\sin(2\theta)}{2}\right]_0^{2\pi},$$

$$\boxed{A = 22\pi.}$$