Math 210, Exam 2, Fall 2001
Problem 1 Solution

1. Let $f(x,y) = xe^{xy^2}$.

   a) Find the directional derivative of f at the point $P = (0,2)$ in the direction of the vector $\vec{\mathbf{v}} = \langle 3,4\rangle$.

   b) In what direction does f increase fastest at the point $P = (0,2)$?

Solution:

a) The value of the directional derivative of f at $P = (0,2)$ in the direction $\vec{\mathbf{v}}$ is:

$$D_{\hat{\mathbf{u}}}f(0,2) = \vec{\nabla} f(0,2) \cdot \vec{\mathbf{v}}$$

where $\hat{\mathbf{u}}$ is a unit vector in the direction of $\vec{\mathbf{v}}$. That is,

$$\hat{\mathbf{u}} = \frac{1}{|\vec{\mathbf{v}}|}\,\vec{\mathbf{v}} = \frac{1}{5}\langle 3,4\rangle$$

The gradient of f is:

$$\vec{\nabla} f = \langle f_x, f_y\rangle$$
$$\vec{\nabla} f = \left\langle e^{xy^2} + xy^2e^{xy^2}, 2x^2ye^{xy^2}\right\rangle$$

and its value at the point $P = (0,2)$ is:

$$\vec{\nabla} f(0,2) = \langle 1,0\rangle$$

Thus, the directional derivative is:

$$\begin{aligned} D_{\hat{\mathbf{u}}}f(0,2) &= \vec{\nabla} f(0,2) \cdot \vec{\mathbf{v}} \\ &= \langle 1,0\rangle \cdot \frac{1}{5}\langle 3,4\rangle \\ &= \boxed{\frac{3}{5}} \end{aligned}$$

b) The direction in which f increases fastest at $P = (0,2)$ is the direction of **steepest ascent**:

$$\begin{aligned} \hat{\mathbf{u}} &= \frac{1}{\left|\vec{\nabla} f(0,2)\right|}\,\vec{\nabla} f(0,2) \\ &= \frac{1}{|1,0|}\,\langle 1,0\rangle \\ &= \boxed{\langle 1,0\rangle} \end{aligned}$$

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Problem 2 Solution

2. Find the maximum and minimum value of $f(x, y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 9$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that $x^2 + y^2 = 9$ is compact and that f is continuous at all points on the circle, guaranteeing the existence of absolute extrema of f. Then, let $g(x, y) = x^2 + y^2 = 9$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad g(x, y) = 9$$

which, when applied to our functions f and g, give us:

$$4x = \lambda\,(2x) \tag{1}$$
$$2y = \lambda\,(2y) \tag{2}$$
$$x^2 + y^2 = 9 \tag{3}$$

We begin by noting that Equation (1) gives us:

$$\begin{aligned} 4x &= \lambda(2x) \\ 4x - \lambda(2x) &= 0 \\ 2x(2 - \lambda) &= 0 \end{aligned}$$

From this equation we either have $x = 0$ or $\lambda = 2$. Let's consider each case separately.

Case 1: Let $x = 0$. We find the corresponding y-values using Equation (3).

$$\begin{aligned} x^2 + y^2 &= 9 \\ 0^2 + y^2 &= 9 \\ y^2 &= 9 \\ y &= \pm 3 \end{aligned}$$

Thus, the points of interest are $(0, 3)$ and $(0, -3)$.

Case 2: Let $\lambda = 2$. Plugging this into Equation (2) we get:

$$\begin{aligned} 2y &= \lambda(2y) \\ 2y &= 2(2y) \\ -2y &= 0 \\ y &= 0 \end{aligned}$$

We find the corresponding x-values using Equation (3).

$$\begin{aligned} x^2 + y^2 &= 9 \\ x^2 + 0^2 &= 9 \\ x^2 &= 9 \\ x &= \pm 3 \end{aligned}$$

Thus, the points of interest are $(3, 0)$ and $(-3, 0)$.

We now evaluate $f(x, y) = 2x^2 + y^2$ at each point of interest obtained in Cases 1 and 2.

$$f(0,3) = 9$$
$$f(0,-3) = 9$$
$$f(3,0) = 18$$
$$f(3,0) = 18$$

From the values above we observe that f attains an absolute maximum of 18 and an absolute minimum of 9.

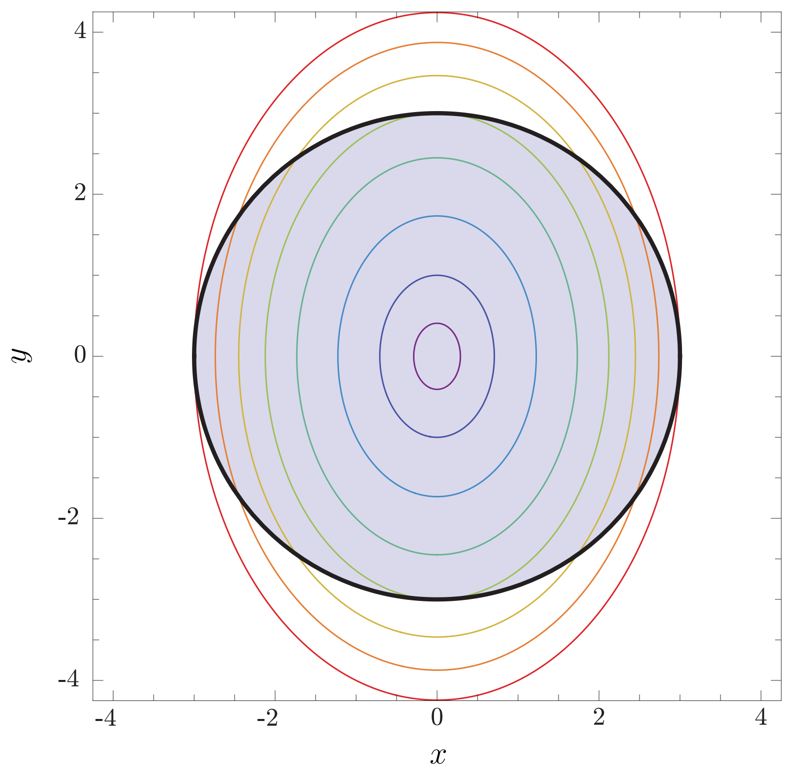

Figure 1: Shown in the figure are the level curves of $f(x, y) = 2x^2 + y^2$ and the circle $x^2 + y^2 = 9$ (thick, black curve). Darker colors correspond to smaller values of $f(x, y)$. Notice that (1) the level curve $f(x, y) = 9$ is tangent to the circle at $(0, 3)$ and $(0, -3)$ which correspond to absolute minima and (2) the level curve $f(x, y) = 18$ is tangent to the circle at $(3, 0)$ and $(-3, 0)$ which correspond to absolute maxima.

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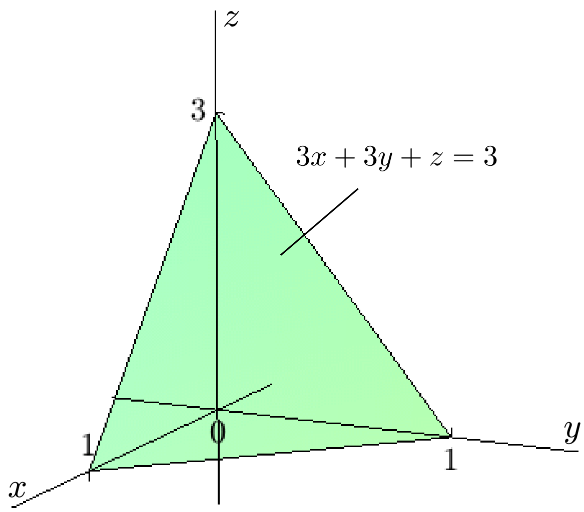
Problem 3 Solution

3. Find the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 3y + z = 3$.

Solution: The region is plotted below.



We use a triple integral to compute the volume. The formula we use is:

$$V = \iiint_D 1\, dV$$

The region D is bounded below by the plane $z = 0$ and above by the plane $z = 3-3x-3y$. The projection of D onto the xy-plane is the triangular region $\{(x, y) \,:\, 0 \leq y \leq 1-x,\ 0 \leq x \leq 1\}$. The line $y = 1 - x$ is the intersection of the plane $z = 3 - 3x - 3y$ and the plane $z = 0$.

$$\begin{aligned} z &= z \\ 0 &= 3 - 3x - 3y \\ 3y &= 3 - 3x \\ y &= 1 - x \end{aligned}$$

Using the order of integration $dz\,dy\,dx$, the volume is:

$$\begin{aligned}
V = \iiint_D 1\,dV &= \int_0^1 \int_0^{1-x} \int_0^{3-3x-3y} 1\,dz\,dy\,dx \\
&= \int_0^1 \int_0^{1-x} \Big[z\Big]_0^{3-3x-3y} dy\,dx \\
&= \int_0^1 \int_0^{1-x} (3-3x-3y)\;dy\,dx \\
&= \int_0^1 \left[3y - 3xy - \frac{3}{2}y^2\right]_0^{1-x} dx \\
&= \int_0^1 \left[3(1-x) - 3x(1-x) - \frac{3}{2}(1-x)^2\right] dx \\
&= \int_0^1 \left(3 - 3x - 3x + 3x^2 - \frac{3}{2} + 3x - \frac{3}{2}x^2\right) dx \\
&= \int_0^1 \left(\frac{3}{2}x^2 - 3x + \frac{3}{2}\right) dx \\
&= \left[\frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{3}{2}x\right]_0^1 \\
&= \boxed{\frac{1}{2}}
\end{aligned}$$

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Problem 4 Solution

4. Let L be a square lamina with verties at $(0,0)$, $(4,0)$, $(4,4)$, and $(0,4)$. Its density function is $\rho(x,y) = y$. Is it possible for the point $(2,1)$ to be the center of mass of L?

Solution: By definition, the center of mass coordinates are:

$$x_{cm} = \frac{M_y}{M} = \frac{\iint_L x\rho(x,y)\,dA}{\iint_L \rho(x,y)\,dA}$$
$$y_{cm} = \frac{M_x}{M} = \frac{\iint_L y\rho(x,y)\,dA}{\iint_L \rho(x,y)\,dA}$$

The numerators in the center of mass formulas are the mass moments and are computed as follows:

$$\begin{aligned}\iint_L x\rho(x,y)\,dA &= \int_0^4 \int_0^4 xy\,dy\,dx \\ &= \int_0^4 x\left[\frac{1}{2}y^2\right]_0^4 dx \\ &= \int_0^4 8x\,dx \\ &= \left[4x^2\right]_0^4 \\ &= 64\end{aligned}$$

$$\begin{aligned}\iint_L y\rho(x,y)\,dA &= \int_0^4 \int_0^4 y^2\,dy\,dx \\ &= \int_0^4 \left[\frac{1}{3}y^3\right]_0^4 dx \\ &= \int_0^4 \frac{64}{3}\,dx \\ &= \left[\frac{64}{3}x\right]_0^4 \\ &= \frac{256}{3}\end{aligned}$$

The denominator in the center of mass formula is the mass of L and is computed as follows:

$$\begin{aligned}\iint_L \rho(x,y)\,dy\,dx &= \int_0^4 \int_0^4 y\,dy\,dx \\ &= \int_0^4 \left[\frac{1}{2}y^2\right]_0^4 dx \\ &= \int_0^4 8\,dx \\ &= \left[8x\right]_0^4 \\ &= 32\end{aligned}$$

Thus, the center of mass coordinates are:

$$x_{cm} = \frac{M_y}{M} = \frac{64}{32} = 2$$
$$y_{cm} = \frac{M_x}{M} = \frac{\frac{256}{3}}{32} = \frac{8}{3}$$

and it is not possible for the point $(2,1)$ to be the center of mass.

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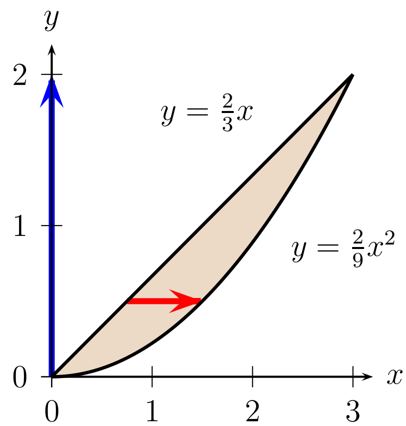
Problem 5 Solution

5. Change the order of integration:

$$\int_0^3 \int_{\frac{2}{9}x^2}^{\frac{2}{3}x} f\, dy\, dx$$

Solution: The region of integration is sketched below:



To change the order of integration we solve the equations $y = \frac{2}{3}x$ and $y = \frac{2}{9}x^2$ for x in terms of y to get:

$$x = \frac{3}{2}y, \qquad x = \sqrt{\frac{9}{2}y}$$

The range of y-values over which the region of integration is defined is $0 \le y \le 2$. Therefore, the integral becomes:

$$\int_0^3 \int_{\frac{2}{9}x^2}^{\frac{2}{3}x} f\, dy\, dx = \int_0^2 \int_{\frac{3}{2}x}^{\sqrt{\frac{9}{2}y}} f\, dx\, dy$$

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Problem 6 Solution

6. Calculate the Jacobian of the transformation:

$$\begin{aligned} x &= u + 2v - 3w \\ y &= 2u - w \\ z &= v \end{aligned}$$

Solution: The Jacobian is the matrix:

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

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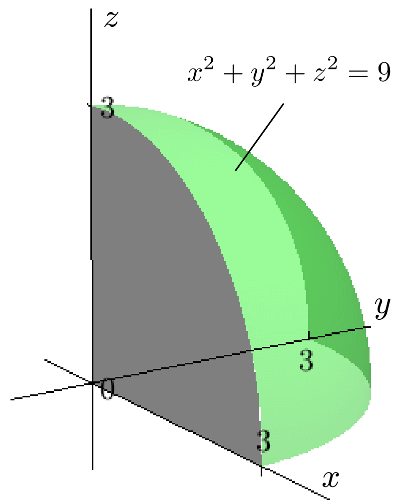
Problem 7 Solution

7. Evaluate:

$$\iiint_R \frac{x}{x^2+y^2}\,dV$$

where R is the region in the first octant bounded by the sphere $x^2+y^2+z^2=9$ and the planes $x=0$, $y=0$ and $z=0$.

Solution: The region is plotted below.



In spherical coordinates, the equation of the sphere is $\rho = 3$. Furthermore, since the region is constrained to the first octant, we know that $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \theta \le \frac{\pi}{2}$. The integrand written in spherical coordinates is:

$$\begin{aligned} f(x,y,z) &= \frac{x}{x^2+y^2} \\ f(\rho,\phi,\theta) &= \frac{\rho\sin\phi\cos\theta}{\rho^2\sin^2\phi} \\ &= \frac{\cos\theta}{\rho\sin\phi} \end{aligned}$$

Using the fact that $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ in spherical coordinates, the value of the integral is:

$$\begin{aligned}
\iiint_R \frac{x}{x^2+y^2}\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \frac{\cos\theta}{\rho\sin\phi}\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho\cos\theta\,d\rho\,d\phi\,d\theta \\
&= \int_0^{\pi/2}\int_0^{\pi/2} \cos\theta\left[\frac{1}{2}\rho^2\right]_0^3\,d\phi\,d\theta \\
&= \frac{9}{2}\int_0^{\pi/2}\int_0^{\pi/2} \cos\theta\,d\phi\,d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} \cos\theta\Big[\phi\Big]_0^{\pi/2}\,d\theta \\
&= \frac{9\pi}{4}\int_0^{\pi/2} \cos\theta\,d\theta \\
&= \frac{9\pi}{4}\Big[\sin\theta\Big]_0^{\pi/2} \\
&= \frac{9\pi}{2}\left[\sin\frac{\pi}{2}-\sin 0\right] \\
&= \boxed{\frac{9\pi}{2}}
\end{aligned}$$