Math 210, Exam 2, Fall 2010
Problem 1 Solution

1. Let $f(x, y) = \frac{1}{3}x^3 + y^2 - xy$. Find all critical points of $f(x, y)$ and classify each as a local maximum, local minimum, or saddle point.

Solution: By definition, an interior point (a, b) in the domain of f is a **critical point** of f if either

(1) $f_x(a, b) = f_y(a, b) = 0$, or

(2) one (or both) of f_x or f_y does not exist at (a, b).

The partial derivatives of $f(x, y) = \frac{1}{3}x^3 + y^2 - xy$ are $f_x = x^2 - y$ and $f_y = 2y - x$. These derivatives exist for all (x, y) in $\mathbb{R}^2$. Thus, the critical points of f are the solutions to the system of equations:

$$f_x = x^2 - y = 0 \tag{1}$$
$$f_y = 2y - x = 0 \tag{2}$$

Solving Equation (1) for y we get:

$$y = x^2 \tag{3}$$

Substituting this into Equation (2) and solving for x we get:

$$\begin{aligned} 2y - x &= 0 \\ 2\left(x^2\right) - x &= 0 \\ x(2x - 1) &= 0 \\ \iff \quad x = 0 \quad \text{or} \quad x &= \frac{1}{2} \end{aligned}$$

We find the corresponding y-values using Equation (3): $y = x^2$.

- If $x = 0$, then $y = 0^2 = 0$.
- If $x = \frac{1}{2}$, then $y = (\frac{1}{2})^2 = \frac{1}{4}$.

Thus, the critical points are $\boxed{(0, 0)}$ and $\boxed{(\frac{1}{2}, \frac{1}{4})}$.

We now use the **Second Derivative Test** to classify the critical points. The second derivatives of f are:

$$f_{xx} = 2x, \quad f_{yy} = 2, \quad f_{xy} = -1$$

The discriminant function $D(x, y)$ is then:

$$\begin{aligned} D(x, y) &= f_{xx}f_{yy} - f_{xy}^2 \\ D(x, y) &= (2x)(2) - (-1)^2 \\ D(x, y) &= 4x - 1 \end{aligned}$$

The values of $D(x, y)$ at the critical points and the conclusions of the Second Derivative Test are shown in the table below.

(a,b)	$D(a,b)$	$f_{xx}(a,b)$	**Conclusion**
$(0,0)$	-1	0	Saddle Point
$(\frac{1}{2},\frac{1}{4})$	1	1	Local Minimum

Recall that (a,b) is a saddle point if $D(a,b) < 0$ and that (a,b) corresponds to a local minimum of f if $D(a,b) > 0$ and $f_{xx}(a,b) > 0$.

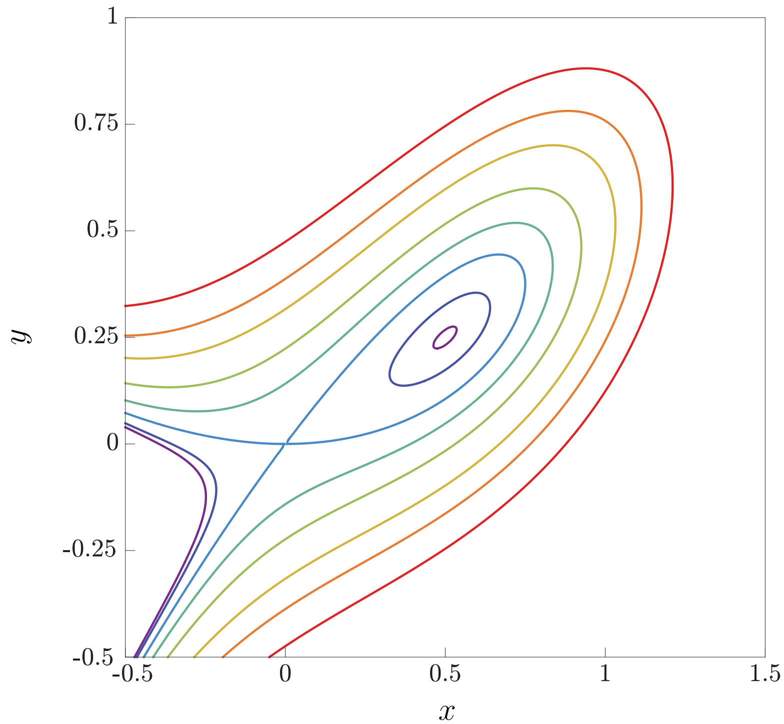

Figure 1: Pictured above are level curves of $f(x,y)$. Darker colors correspond to smaller values of $f(x,y)$. It is apparent that $(0,0)$ is a saddle point and $(\frac{1}{2},\frac{1}{4})$ corresponds to a local minimum.

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Problem 2 Solution

2. Find the minimum and maximum of the function $f(x, y, z) = x - y - z$ on the ellipsoid $R = \left\{ (x, y, z) \;\middle|\; \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \right\}$.

Solution: We find the minimum and maximum using the method of **Lagrange Multipliers**. First, we recognize that R is compact which guarantees the existence of absolute extrema of f. Then, let $g(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1$. We look for solutions to the following system of equations:

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad f_z = \lambda g_z, \quad g(x, y, z) = 1$$

which, when applied to our functions f and g, give us:

$$1 = \lambda \left(\frac{2x}{4} \right) \tag{1}$$

$$-1 = \lambda \left(\frac{2y}{9} \right) \tag{2}$$

$$-1 = \lambda \left(\frac{2z}{3} \right) \tag{3}$$

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} = 1 \tag{4}$$

To solve the system of equations, we first solve Equations (1)-(3) for the variables x, y, and z in terms of λ to get:

$$x = \frac{4}{2\lambda}, \quad y = -\frac{9}{2\lambda}, \quad z = -\frac{3}{2\lambda} \tag{5}$$

We then plug Equations (5) into Equation (4) and simplify.

$$\begin{aligned} \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{3} &= 1 \\ \frac{(\frac{4}{2\lambda})^2}{4} + \frac{(-\frac{9}{2\lambda})^2}{9} + \frac{-(\frac{3}{2\lambda})^2}{3} &= 1 \\ \frac{\frac{16}{4\lambda^2}}{4} + \frac{\frac{81}{4\lambda^2}}{9} + \frac{\frac{9}{4\lambda^2}}{3} &= 1 \end{aligned}$$

At this point we multiply both sides of the equation by $4\lambda^2$ to get:

$$\begin{aligned} 4\lambda^2 \left(\frac{\frac{16}{4\lambda^2}}{4} + \frac{\frac{81}{4\lambda^2}}{9} + \frac{\frac{9}{4\lambda^2}}{3} \right) &= 4\lambda^2(1) \\ \frac{16}{4} + \frac{81}{9} + \frac{9}{3} &= 4\lambda^2 \\ 4 + 9 + 3 &= 4\lambda^2 \\ \lambda^2 &= 4 \\ \lambda &= \pm 2 \end{aligned}$$

- When $\lambda = 2$, Equations (5) give us the first candidate for the location of an extreme value:
$$x = 1, \quad y = -\frac{9}{4}, \quad z = -\frac{3}{4}$$
- When $\lambda = -2$, Equations (5) give us the second candidate for the location of an extreme value:
$$x = -1, \quad y = \frac{9}{4}, \quad z = \frac{3}{4}$$

Evaluating $f(x, y, z)$ at these points we find that:

$$f\left(1, -\frac{9}{4}, -\frac{3}{4}\right) = 1 - \left(-\frac{9}{4}\right) - \left(-\frac{3}{4}\right) = 4$$
$$f\left(-1, \frac{9}{4}, \frac{3}{4}\right) = -1 - \left(\frac{9}{4}\right) - \left(\frac{3}{4}\right) = -4$$

Therefore, the absolute maximum value of f on R is 4 and the absolute minimum of f on R is -4.

Note: The level surfaces $f(x, y, z) = 4$ and $f(x, y, z) = -4$ are planes tangent to the ellipsoid at the critical points.

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Problem 3 Solution

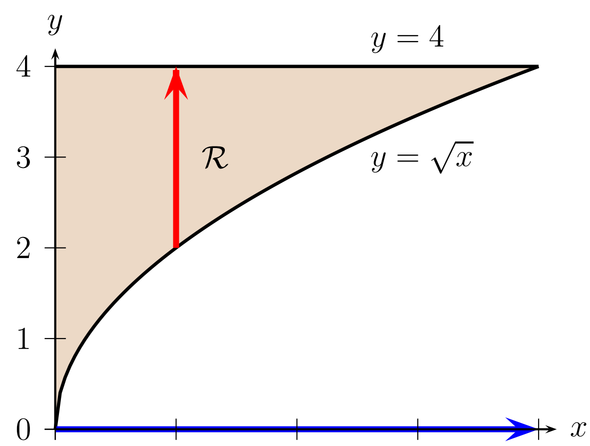
3. Consider the double integral: $\displaystyle\int_0^4 \int_0^{y^2} \frac{x^3}{4-\sqrt{x}}\,dx\,dy$.

(a) Sketch the region of integration.

(b) Change the order of integration.

(c) Evaluate the integral from part (b).

Solution:



(a)

(b) From the figure we see that the region $\mathcal{R}$ is bounded above by $y = 4$ and below by $y = \sqrt{x}$ (obtained by solving $x = y^2$ for y in terms of x). The projection of $\mathcal{R}$ onto the x-axis is the interval $0 \le x \le 16$. Upon changing the order of integration we get the double integral:

$$\int_0^{16} \int_{\sqrt{x}}^{4} \frac{x^3}{4-\sqrt{x}}\,dy\,dx$$

(c) The integral from part (b) is evaluated as follows:

$$\begin{aligned}
\int_0^{16} \int_{\sqrt{x}}^{4} \frac{x^3}{4-\sqrt{x}}\,dy\,dx &= \int_0^{16} \frac{x^3}{4-\sqrt{x}} \Big[y\Big]_{\sqrt{x}}^{4} dx \\
&= \int_0^{16} \frac{x^3}{4-\sqrt{x}} \left(4-\sqrt{x}\right) dx \\
&= \int_0^{16} x^3\,dx \\
&= \left[\frac{1}{4}x^4\right]_0^{16} \\
&= \frac{1}{4}(16)^4 \\
&= \boxed{16384}
\end{aligned}$$

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Problem 4 Solution

4. For the vector field $\overrightarrow{\mathbf{F}} = \langle yx^2, y^2 \rangle$, find the value of $\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}}$ where $\mathcal{C}$ is the portion of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$.

Solution: We evaluate the vector line integral using the formula:

$$\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} = \int_a^b \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{r}}'(t)\, dt$$

A parameterization of $\mathcal{C}$ is $\overrightarrow{\mathbf{r}}(t) = \langle t, t^2 \rangle$, $0 \le t \le 1$. The derivative is $\overrightarrow{\mathbf{r}}'(t) = \langle 1, 2t \rangle$. Using the fact that $x = t$ and $y = t^2$ from the parameterization, the vector field $\overrightarrow{\mathbf{F}}$ written in terms of t is:

$$\overrightarrow{\mathbf{F}} = \left\langle yx^2, y^2 \right\rangle = \left\langle (t^2)(t)^2, (t^2)^2 \right\rangle = \left\langle t^4, t^4 \right\rangle$$

Thus, the value of the line integral is:

$$\begin{aligned}
\int_{\mathcal{C}} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{s}} &= \int_a^b \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{r}}'(t)\, dt \\
&= \int_0^1 \left\langle t^4, t^4 \right\rangle \cdot \langle 1, 2t \rangle \, dt \\
&= \int_0^1 \left(t^4 + 2t^5 \right) \, dt \\
&= \left[\frac{1}{5}t^5 + \frac{1}{3}t^6 \right]_0^1 \\
&= \left[\frac{1}{5}(1)^5 + \frac{1}{3}(1)^6 \right] - \left[\frac{1}{5}(0)^5 + \frac{1}{3}(0)^6 \right] \\
&= \boxed{\frac{8}{15}}
\end{aligned}$$

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Problem 5 Solution

5. Consider the vector field $\overrightarrow{\mathbf{F}} = \langle ax^2y + 8xy^2 - 4, bx^2y - 2x^3 - 1 \rangle$ where a and b are constants.

(a) Find the values of a and b for which $\overrightarrow{\mathbf{F}}$ is conservative.

(b) For the values of a and b from part (a), find a potential function $\varphi(x, y)$ such that $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$.

Solution:

(a) In order for the vector field $\overrightarrow{\mathbf{F}} = \langle f(x, y), g(x, y) \rangle$ to be conservative, it must be the case that:

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$$

Using $f(x, y) = ax^2y + 8xy^2 - 4$ and $g(x, y) = bx^2y - 2x^3 - 1$ we get:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial g}{\partial x} \\ ax^2 + 16xy &= 2bxy - 6x^2 \\ ax^2 + 6x^2 &= 2bxy - 16xy \\ (a+6)x^2 &= (2b-16)xy \end{aligned}$$

In order for the above equation to be satisfied for all pairs (x, y), it must be the case that $a + 6 = 0$ and $2b - 16 = 0$ which give us $\boxed{a = -6}$ and $\boxed{b = 8}$.

(b) If $\overrightarrow{\mathbf{F}} = \overrightarrow{\nabla}\varphi$, then it must be the case that:

$$\frac{\partial \varphi}{\partial x} = f(x, y) \tag{1}$$

$$\frac{\partial \varphi}{\partial y} = g(x, y) \tag{2}$$

Using $f(x, y) = -6x^2y + 8xy^2 - 4$ and integrating both sides of Equation (1) with respect to x we get:

$$\begin{aligned} \frac{\partial \varphi}{\partial x} &= f(x, y) \\ \frac{\partial \varphi}{\partial x} &= -6x^2y + 8xy^2 - 4 \\ \int \frac{\partial \varphi}{\partial x}\, dx &= \int \left(-6x^2y + 8xy^2 - 4\right)\, dx \end{aligned}$$

$$\varphi(x, y) = -2x^3y + 4x^2y^2 - 4x + h(y) \tag{3}$$

We obtain the function $h(y)$ using Equation (2). Using $g(x, y) = 8x^2y - 2x^3 - 1$ we get the equation:

$$\frac{\partial \varphi}{\partial y} = g(x, y)$$
$$\frac{\partial \varphi}{\partial y} = 8x^2y - 2x^3 - 1$$

We now use Equation (3) to obtain the left hand side of the above equation. Simplifying we get:

$$\begin{aligned}\frac{\partial}{\partial y}\left(-2x^3y + 4x^2y^2 - 4x + h(y)\right) &= 8x^2y - 2x^3 - 1\\ -2x^3 + 8x^2y + h'(y) &= 8x^2y - 2x^3 - 1\\ h'(y) &= -1\end{aligned}$$

Now integrate both sides with respect to y to get:

$$\begin{aligned}\int h'(y)\,dy &= \int -1\,dy\\ h(y) &= -y + C\end{aligned}$$

Letting $C = 0$, we find that a potential function for $\overrightarrow{\mathbf{F}}$ is:

$$\boxed{\varphi(x, y) = -2x^3y + 4x^2y^2 - 4x - y}$$

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Problem 6 Solution

6. Compute the surface area of the part of the paraboloid $z = 4 - x^2 - y^2$ that lies in the region $\{(x, y, z) \,|\, z \geq 3,\ x \geq 0\}$.

Solution: The formula for surface area we will use is:

$$S = \iint_{\mathcal{S}} dS = \iint_{\mathcal{R}} \left| \overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v \right| dA$$

where the function $\overrightarrow{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ with domain $\mathcal{R}$ is a parameterization of the surface $\mathcal{S}$ and the vectors $\overrightarrow{\mathbf{t}}_u = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial u}$ and $\overrightarrow{\mathbf{t}}_v = \frac{\partial \overrightarrow{\mathbf{r}}}{\partial v}$ are the tangent vectors.

We begin by finding a parameterization of the paraboloid. Let $x = u\cos(v)$ and $y = u\sin(v)$, where we define u to be nonnegative. Then,

$$\begin{aligned}
z &= 4 - x^2 - y^2 \\
z &= 4 - (u\cos(v))^2 - (u\sin(v))^2 \\
z &= 4 - u^2\cos^2(v) - u^2\sin^2(v) \\
z &= 4 - u^2
\end{aligned}$$

Thus, we have $\overrightarrow{\mathbf{r}}(u, v) = \langle u\cos(v), u\sin(v), 4 - u^2\rangle$. To find the domain $\mathcal{R}$, we must interpret the inequalities $z \geq 0$ and $x \geq 0$ in terms of the new variables u and v. From the first inequality we find that:

$$\begin{aligned}
z &\geq 0 \\
4 - u^2 &\geq 0 \\
u^2 &\leq 4 \\
0 \leq u &\leq 2
\end{aligned}$$

noting that, by definition, u must be nonnegative. From the second inequality we find that:

$$\begin{aligned}
x &\geq 0 \\
u\cos(v) &\geq 0 \\
\cos(v) &\geq 0 \\
-\frac{\pi}{2} \leq v &\leq \frac{\pi}{2}
\end{aligned}$$

noting that $\cos(v) \geq 0$ implies that v is an angle in either Quadrant I or IV. Therefore, a parameterization of $\mathcal{S}$ is:

$$\begin{aligned}
\overrightarrow{\mathbf{r}}(u, v) &= \left\langle u\cos(v), u\sin(v), 4 - u^2 \right\rangle, \\
\mathcal{R} &= \left\{ (u, v) \,\middle|\, 0 \leq u \leq 2,\ -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \right\}
\end{aligned}$$

The tangent vectors $\overrightarrow{\mathbf{t}}_u$ and $\overrightarrow{\mathbf{t}}_v$ are then:

$$\begin{aligned}
\overrightarrow{\mathbf{t}}_u &= \frac{\partial \overrightarrow{\mathbf{r}}}{\partial u} = \langle \cos(v), \sin(v), -2u \rangle \\
\overrightarrow{\mathbf{t}}_v &= \frac{\partial \overrightarrow{\mathbf{r}}}{\partial v} = \langle -u\sin(v), u\cos(v), 0 \rangle
\end{aligned}$$

The cross product of these vectors is:

$$\begin{aligned}
\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix} \\
&= 2u^2\cos(v)\,\hat{\mathbf{i}} + 2u^2\sin(v)\,\hat{\mathbf{j}} + u\,\hat{\mathbf{k}} \\
&= \left\langle 2u^2\cos(v), 2u^2\sin(v), u \right\rangle
\end{aligned}$$

The magnitude of the cross product is:

$$\begin{aligned}
\left|\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v\right| &= \sqrt{\left(2u^2\cos(v)\right)^2 + \left(2u^2\sin(v)\right)^2 + u^2} \\
&= \sqrt{4u^4\cos^2(v) + 4u^4\sin^2(v) + u^2} \\
&= \sqrt{4u^4 + u^2} \\
&= u\sqrt{4u^2+1}
\end{aligned}$$

We can now compute the surface area.

$$\begin{aligned}
S &= \iint_{\mathcal{R}} \left|\overrightarrow{\mathbf{t}}_u \times \overrightarrow{\mathbf{t}}_v\right| \, dA \\
&= \int_0^2 \int_{-\pi/2}^{\pi/2} u\sqrt{4u^2+1}\, dv\, du \\
&= \int_0^2 u\sqrt{4u^2+1} \Big[v \Big]_{-\pi/2}^{\pi/2} du \\
&= \int_0^2 u\sqrt{4u^2+1} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right] du \\
&= \int_0^2 \pi u\sqrt{4u^2+1}\, du \\
&= \left[\frac{\pi}{12}\left(4u^2+1\right)^{3/2} \right]_0^2 \\
&= \left[\frac{\pi}{12}\left(4(2)^2+1\right)^{3/2} \right] - \left[\frac{\pi}{12}\left(4(0)^2+1\right)^{3/2} \right] \\
&= \frac{\pi}{12}(17)^{3/2} - \frac{\pi}{12}(1)^{3/2} \\
&= \boxed{\frac{\pi}{12}\left(17\sqrt{17}-1\right)}
\end{aligned}$$