1. Find an equation of the plane passing through the following three points: $P = (2, -1, 4)$, $Q = (1, 1, -1)$, $R = (-4, 1, 1)$.

Solution: Let $\vec{u} = \overrightarrow{PQ} = (-1, 2, -5)$ and $\vec{v} = \overrightarrow{QR} = (-5, 0, 2)$. The cross product of $\vec{u}$ and $\vec{v}$ results in a vector normal to the plane containing $P$, $Q$, and $R$.

$$\vec{u} \times \vec{v} = \langle 4, 27, 10 \rangle.$$

A plane containing a point $(x_0, y_0, z_0)$ with normal vector $\langle a, b, c \rangle$ has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Using $P = (2, -1, 4)$ as a point in the plane we have

$$4(x - 2) + 27(y + 1) + 10(z - 4) = 0.$$
2. Let the position vector be given by \( \mathbf{r}(t) = 2t^3 \mathbf{i} + (t^2 - t) \mathbf{j} - 8t \mathbf{k} \). Find the angle between the velocity and acceleration vectors at time \( t = 0 \).

**Solution:** The velocity and acceleration vectors are the first and second derivatives of \( \mathbf{r}(t) \), respectively.

\[
\mathbf{r}'(t) = \langle 6t^2, 2t - 1, -8 \rangle, \quad \mathbf{r}''(t) = \langle 12t, 2, 0 \rangle.
\]

The vectors evaluated at \( t = 0 \) are

\[
\mathbf{r}'(0) = \langle 0, -1, -8 \rangle, \quad \mathbf{r}''(0) = \langle 0, 2, 0 \rangle.
\]

The angle between two vectors can be computed via the dot product. That is,

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \cdot ||\mathbf{v}||}.
\]

Letting \( \mathbf{u} = \langle 0, -1, -8 \rangle \) and \( \mathbf{v} = \langle 0, 2, 0 \rangle \) we find that

\[
\cos \theta = \frac{-2}{2\sqrt{65}} \iff \theta = \arccos \left( -\frac{1}{\sqrt{65}} \right).
\]
3. Let $z = \sin x \cos y$, where $x = s + t$, $y = s - t$. Use the chain rule to compute the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: The Chain Rule formulas for a function $z = z(x,y)$ where $x = x(s,t)$ and $y = y(s,t)$ are

$$
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s},
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}.
$$

Using the fact that $z = \sin x \cos y$ we have

$$
\frac{\partial z}{\partial x} = \cos x \cos y, \quad \frac{\partial z}{\partial y} = -\sin x \sin y.
$$

Furthermore, since $x = s + t$ and $y = s - t$ we have

$$
\frac{\partial x}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = 1, \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial y}{\partial t} = -1.
$$

Using the Chain Rule formulas we get

$$
\frac{\partial z}{\partial s} = \cos x \cos y - \sin x \sin y = \cos(x + y),
\frac{\partial z}{\partial t} = \cos x \cos y + \sin x \sin y = \cos(x - y).
$$

Using the fact that $x + y = 2s$ and $x - y = 2t$ we arrive at our answers in terms of $s$ and $t$

$$
\frac{\partial z}{\partial s} = \cos(2s), \quad \frac{\partial z}{\partial t} = \cos(2t).
$$
4. Let $f(x, y) = \ln(2x + y)$.

(a) Write the equation of the tangent plane to the graph of $f(x, y)$ at $(-1, 3)$.

(b) Use part (a) to estimate $f(-1.1, 2.9)$.

Solution:

(a) For a function written explicitly as a function of $x$ and $y$ we have the following formula for the tangent plane at the point $(x_0, y_0)$:

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The first partial derivatives of $f(x, y)$ are

$$f_x = \frac{2}{2x + y}, \quad f_y = \frac{1}{2x + y}.$$  

The values of $f$ and the first partial derivatives of $f$ at $(-1, 3)$ are

$$f(-1, 3) = 0, \quad f_x(-1, 3) = 2, \quad f_y(-1, 3) = 1.$$  

Thus, an equation for the tangent plane at $(-1, 3)$ is

$$z = 2(x + 1) + (y - 3).$$

(b) An estimate for $f(a, b)$ may be taken as the value of $L(a, b)$, the linearization of $f(x, y)$ at a point near $(a, b)$. Since the linearization and the tangent plane are one in the same, we know that

$$L(x, y) = 2(x + 1) + (y - 3).$$

Evaluating $L$ at $(-1.1, 2.9)$ we get

$$L(-1.1, 2.9) = 2(-1.1 + 1) + (2.9 - 3) = -0.3.$$
5. Evaluate the triple integral 
\[ \iiint_D y \, dV, \]
where \( D \) is the region inside the cylinder \( x^2 + y^2 = 9 \) above the plane \( z = x - 2 \) and below the plane \( z = 2 - x \).

**Solution:** The region \( D \) can be described in Cartesian coordinates as follows:
\[
D = \left\{ (x, y, z) : x - 2 \leq x \leq 2 - x, \quad -\sqrt{9 - x^2} \leq y \leq \sqrt{9 - x^2}, \quad -3 \leq x \leq 2 \right\}
\]
The inequalities that describe \( x \) and \( y \) are determined by the projection of \( D \) onto the \( xy \)-plane, which is pictured below.

Thus, the integral is set up and evaluated as follows:
\[
\iiint_D y \, dV = \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x-2}^{2-x} y \, dz \, dy \, dx,
\]
\[
= \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y \left[ \frac{2}{2} x - 2 \right] dy \, dx,
\]
\[
= \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y(4 - 2x) \, dy \, dx,
\]
\[
= \int_{-3}^{2} (4 - 2x) \left[ \frac{1}{2} y^2 \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \, dx,
\]
\[
= \int_{-3}^{2} (4 - 2x) \left[ \frac{1}{2}(9 - x^2) - \frac{1}{2}(9 - x^2) \right] \, dx,
\]
\[
= 0.
\]
6. Find a potential function for the vector field \( \mathbf{F}(x, y) = xe^{x^2+y^2} \mathbf{i} + ye^{x^2+y^2} \mathbf{j} \). Compute the line integral of \( \mathbf{F} \) along any path from \((0, 1)\) to \((1, 2)\).

**Solution:** By inspection, a potential function for \( \mathbf{F} \) is

\[
\varphi(x, y) = \frac{1}{2} e^{x^2+y^2}.
\]

Using the Fundamental Theorem of Line Integrals, the line integral of \( \mathbf{F} \) along any path from \((0, 1)\) to \((1, 2)\) has the value

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(1, 2) - \varphi(0, 1) = \frac{1}{2} e^5 - \frac{1}{2} e^1 = \frac{1}{2} e(e^4 - 1).
\]
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7. Let $R = \{(x, y) : x^2 \leq y \leq x\}$. Compute the following integral, using Green’s theorem or otherwise

\[ \oint_C \mathbf{F} \cdot d\mathbf{r}, \]

where $\mathbf{F} = x^3 \mathbf{i} + xy^2 \mathbf{j}$, and $C$ is a counterclockwise oriented boundary of $R$.

**Solution:** Using Green’s Theorem we have

\[
\begin{align*}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_R \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA, \\
&= \int_0^1 \int_{x^2}^x \left( \frac{\partial}{\partial x} xy^2 - \frac{\partial}{\partial y} x^3 \right) dy \, dx, \\
&= \int_0^1 \int_{x^2}^x y^2 \, dy \, dx, \\
&= \int_0^1 \left[ \frac{1}{3} y^3 \right]_{x^2}^x \, dx, \\
&= \frac{1}{3} \int_0^1 \left( x^3 - x^6 \right) \, dx, \\
&= \frac{1}{3} \left[ \frac{1}{4} x^4 - \frac{1}{7} x^7 \right]_0^1, \\
&= \frac{1}{3} \left( \frac{1}{4} - \frac{1}{7} \right), \\
&= \frac{1}{28}.
\end{align*}
\]
8. Consider the region \( R = \{(x, y) : x + y \geq 0, y \leq 0, x \leq 1\} \) and the transformation
\[
T : u = x + y, \ v = x.
\]

(a) Compute the Jacobian \( J(u, v) \).

(b) Find the image of \( R \) in the \( uv \)-plane under the transformation \( T \).

(c) Using (a) and (b) evaluate
\[
\int\int_{R} x^3 \sqrt{x+y} \, dA.
\]

Solution:

(a) The Jacobian of the transformation is
\[
J(u, v) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.
\]

(b) The region \( R \) is a triangle with vertices at \((0, 0)\), \((1, 0)\), and \((1, -1)\). Since \( T \) is a linear transformation and the boundary of \( R \) consists of line segments, we know that the image of \( R \) may be determined by finding the images of the vertices of \( R \).
\[
T(0, 0) = (0 + 0, 0) = (0, 0)
\]
\[
T(1, 0) = (1 + 0, 1) = (1, 1)
\]
\[
T(1, -1) = (1 - 1, 1) = (0, 1)
\]

Thus, the image of \( R \) is the triangular region with vertices at \((0, 0)\), \((1, 1)\), and \((0, 1)\), i.e.
\[
D = \text{Image}(R) = \{(u, v) : 0 \leq u \leq v, \ 0 \leq v \leq 1\}
\]

(c) The Change of Variables formula for computing a double integral is
\[
\int\int_{R} f(x, y) \, dA = \int\int_{D} f(x(u, v), y(u, v))|J(u, v)| \, du \, dv
\]
Since \( f(x, y) = x^3 \sqrt{x+y} \) we have
\[
f(x(u, v), y(u, v)) = v^3 \sqrt{u}.
\]
Thus, the integral has the value

\[
\int \int_{R} f(x, y) \, dA = \int \int_{D} f(x(u, v), y(u, v)) \vert J(u, v) \vert \, du \, dv
\]

\[
= \int_{0}^{1} \int_{0}^{v} v^{3} \sqrt{u} \vert - 1 \vert \, du \, dv,
\]

\[
= \int_{0}^{1} v^{3} \left[ \frac{2}{3} u^{3/2} \right]_{0}^{v} \, dv,
\]

\[
= \frac{2}{3} \int_{0}^{1} v^{3} \cdot v^{3/2} \, dv,
\]

\[
= \frac{2}{3} \int_{0}^{1} v^{9/2} \, dv,
\]

\[
= \frac{2}{3} \left[ \frac{2}{11} v^{11/2} \right]_{0}^{1},
\]

\[
= \frac{4}{33}
\]