Math 210, Final Exam, Fall 2011
Problem 1 Solution

1. Find an equation of the plane passing through the following three points: $P = (2, -1, 4)$, $Q = (1, 1, -1)$, $R = (-4, 1, 1)$.

Solution: Let $\overrightarrow{\mathbf{u}} = \overrightarrow{PQ} = \langle -1, 2, -5 \rangle$ and $\overrightarrow{\mathbf{v}} = \overrightarrow{QR} = \langle -5, 0, 2 \rangle$. The cross product of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$ results in a vector normal to the plane containing P, Q, and R.

$$\overrightarrow{\mathbf{u}} \times \overrightarrow{\mathbf{v}} = \langle 4, 27, 10 \rangle .$$

A plane containing a point (x_0, y_0, z_0) with normal vector $\langle a, b, c \rangle$ has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Using $P = (2, -1, 4)$ as a point in the plane we have

$$\boxed{4(x - 2) + 27(y + 1) + 10(z - 4) = 0.}$$

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Problem 2 Solution

2. Let the position vector be given by $\overrightarrow{\mathbf{r}}(t) = 2t^3\,\hat{\mathbf{i}} + (t^2 - t)\,\hat{\mathbf{j}} - 8t\,\hat{\mathbf{k}}$. Find the angle between the velocity and acceleration vectors at time $t = 0$.

Solution: The velocity and acceleration vectors are the first and second derivatives of $\overrightarrow{\mathbf{r}}(t)$, respectively.

$$\overrightarrow{\mathbf{r}}'(t) = \left\langle 6t^2, 2t - 1, -8 \right\rangle, \qquad \overrightarrow{\mathbf{r}}''(t) = \langle 12t, 2, 0 \rangle .$$

The vectors evaluated at $t = 0$ are

$$\overrightarrow{\mathbf{r}}'(0) = \langle 0, -1, -8 \rangle, \qquad \overrightarrow{\mathbf{r}}''(0) = \langle 0, 2, 0 \rangle .$$

The angle between two vectors can be computed via the dot product. That is,

$$\cos\theta = \frac{\overrightarrow{\mathbf{u}} \bullet \overrightarrow{\mathbf{v}}}{||\overrightarrow{\mathbf{u}}||\,||\overrightarrow{\mathbf{v}}||}.$$

Letting $\overrightarrow{\mathbf{u}} = \langle 0, -1, -8 \rangle$ and $\overrightarrow{\mathbf{v}} = \langle 0, 2, 0 \rangle$ we find that

$$\cos\theta = \frac{-2}{2\sqrt{65}} \quad \Longleftrightarrow \quad \boxed{\theta = \arccos\left(-\frac{1}{\sqrt{65}}\right).}$$

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Problem 3 Solution

3. Let $z = \sin x \cos y$, where $x = s + t$, $y = s - t$. Use the chain rule to compute the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution: The Chain Rule formulas for a function $z = z(x, y)$ where $x = x(s, t)$ and $y = y(s, t)$ are

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s},$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Using the fact that $z = \sin x \cos y$ we have

$$\frac{\partial z}{\partial x} = \cos x \cos y, \qquad \frac{\partial z}{\partial y} = -\sin x \sin y.$$

Furthermore, since $x = s + t$ and $y = s - t$ we have

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial x}{\partial t} = 1, \quad \frac{\partial y}{\partial s} = 1, \quad \frac{\partial y}{\partial t} = -1.$$

Using the Chain Rule formulas we get

$$\frac{\partial z}{\partial s} = \cos x \cos y - \sin x \sin y = \cos(x + y),$$
$$\frac{\partial z}{\partial t} = \cos x \cos y + \sin x \sin y = \cos(x - y).$$

Using the fact that $x + y = 2s$ and $x - y = 2t$ we arrive at our answers in terms of s and t

$$\boxed{\frac{\partial z}{\partial s} = \cos(2s), \qquad \frac{\partial z}{\partial t} = \cos(2t).}$$

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Problem 4 Solution

4. Let $f(x,y) = \ln(2x+y)$.

(a) Write the equation of the tangent plane to the graph of $f(x,y)$ at $(-1,3)$.

(b) Use part (a) to estimate $f(-1.1, 2.9)$.

Solution:

(a) For a function written explicitly as a function of x and y we have the following formula for the tangent plane at the point (x_0, y_0):

$$z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The first partial derivatives of $f(x,y)$ are

$$f_x = \frac{2}{2x+y}, \qquad f_y = \frac{1}{2x+y}.$$

The values of f and the first partial derivatives of f at $(-1,3)$ are

$$f(-1,3) = 0, \qquad f_x(-1,3) = 2, \qquad f_y(-1,3) = 1.$$

Thus, an equation for the tangent plane at $(-1,3)$ is

$$\boxed{z = 2(x+1) + (y-3).}$$

(b) An estimate for $f(a,b)$ may be taken as the value of $L(a,b)$, the linearization of $f(x,y)$ at a point near (a,b). Since the linearization and the tangent plane are one in the same, we know that

$$L(x,y) = 2(x+1) + (y-3).$$

Evaluating L at $(-1.1, 2.9)$ we get

$$\boxed{L(-1.1, 2.9) = 2(-1.1+1) + (2.9-3) = -0.3.}$$

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Problem 5 Solution

5. Evaluate the triple integral

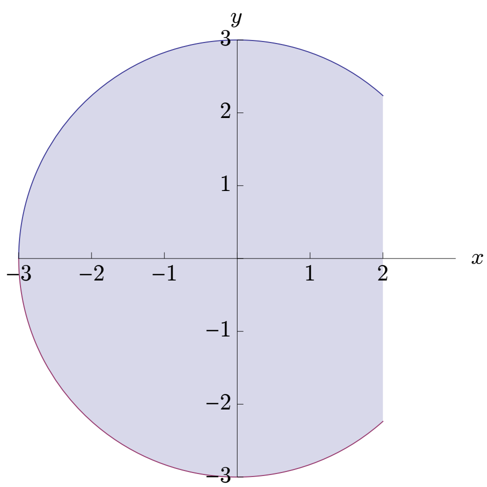
$$\iiint_D y\,dV,$$

where D is the region inside the cylinder $x^2 + y^2 = 9$ above the plane $z = x - 2$ and below the plane $z = 2 - x$.

Solution: The region D can be described in Cartesian coordinates as follows:

$$D = \left\{(x, y, z) \,:\, x - 2 \le x \le 2 - x,\ -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2},\ -3 \le x \le 2\right\}$$

The inequalities that describe x and y are determined by the projection of D onto the xy-plane, which is pictured below.



Thus, the integral is set up and evaluated as follows:

$$\begin{aligned}
\iiint_D y\,dV &= \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x-2}^{2-x} y\,dz\,dy\,dx, \\
&= \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y\Big[z\Big]_{x-2}^{2-x} dy\,dz, \\
&= \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y(4 - 2x)\,dy\,dx, \\
&= \int_{-3}^{2} (4 - 2x) \left[\frac{1}{2}y^2\right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx, \\
&= \int_{-3}^{2} (4 - 2x) \left[\frac{1}{2}(9 - x^2) - \frac{1}{2}(9 - x^2)\right] dx, \\
&= \boxed{0}.
\end{aligned}$$

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Problem 6 Solution

6. Find a potential function for the vector field $\overrightarrow{\mathbf{F}}(x, y) = xe^{x^2+y^2}\,\hat{\mathbf{i}} + ye^{x^2+y^2}\,\hat{\mathbf{j}}$. Compute the line integral of $\overrightarrow{\mathbf{F}}$ along any path from $(0,1)$ to $(1,2)$.

Solution: By inspection, a potential function for $\overrightarrow{\mathbf{F}}$ is

$$\boxed{\varphi(x,y) = \frac{1}{2}e^{x^2+y^2}.}$$

Using the Fundamental Theorem of Line Integrals, the line integral of $\overrightarrow{\mathbf{F}}$ along any path from $(0,1)$ to $(1,2)$ has the value

$$\int_C \overrightarrow{\mathbf{F}} \bullet d\overrightarrow{\mathbf{r}} = \varphi(1,2) - \varphi(0,1) = \frac{1}{2}e^5 - \frac{1}{2}e^1 = \boxed{\frac{1}{2}e(e^4-1)}.$$

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Problem 7 Solution

7. Let $R = \{(x, y) : x^2 \le y \le x\}$. Compute the following integral, using Green's theorem or otherwise

$$\oint_C \overrightarrow{\mathbf{F}} \bullet d\,\overrightarrow{\mathbf{r}},$$

where $\overrightarrow{\mathbf{F}} = x^3\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}}$, and C is a counterclockwise oriented boundary of R.

Solution: Using Green's Theorem we have

$$\begin{aligned}
\oint_C \overrightarrow{\mathbf{F}} \bullet d\,\overrightarrow{\mathbf{r}} &= \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA, \\
&= \int_0^1 \int_{x^2}^{x} \left(\frac{\partial}{\partial x} xy^2 - \frac{\partial}{\partial y} x^3\right) dy\,dx, \\
&= \int_0^1 \int_{x^2}^{x} y^2\,dy\,dx, \\
&= \int_0^1 \left[\frac{1}{3}y^3\right]_{x^2}^{x} dx, \\
&= \frac{1}{3}\int_0^1 \left(x^3 - x^6\right) dx, \\
&= \frac{1}{3}\left[\frac{1}{4}x^4 - \frac{1}{7}x^7\right]_0^1, \\
&= \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right), \\
&= \boxed{\frac{1}{28}}
\end{aligned}$$

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Problem 8 Solution

8. Consider the region $R = \{(x, y) : x + y \geq 0, y \leq 0, x \leq 1\}$ and the transformation

$$T : u = x + y, \ v = x.$$

(a) Compute the Jacobian $J(u, v)$.

(b) Find the image of R in the uv-plane under the transformation T.

(c) Using (a) and (b) evaluate

$$\iint_R x^3\sqrt{x+y}\, dA.$$

Solution:

(a) The Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = \boxed{-1}.$$

(b) The region R is a triangle with vertices at $(0, 0)$, $(1, 0)$, and $(1, -1)$. Since T is a linear transformation and the boundary of R consists of line segments, we know that the image of R may be determined by finding the images of the vertices of R.

$$\begin{aligned} T(0,0) &= (0+0, 0) = (0,0) \\ T(1,0) &= (1+0, 1) = (1,1) \\ T(1,-1) &= (1-1, 1) = (0,1) \end{aligned}$$

Thus, the image of R is the triangular region with vertices at $(0, 0)$, $(1, 1)$, and $(0, 1)$, i.e.

$$D = \text{Image}(R) = \{(u, v) : 0 \leq u \leq v, \ 0 \leq v \leq 1\}$$

(c) The Change of Variables formula for computing a double integral is

$$\iint_R f(x, y)\, dA = \iint_D f(x(u, v), y(u, v))|J(u, v)|\, du\, dv$$

Since $f(x, y) = x^3\sqrt{x+y}$ we have

$$f(x(u, v), y(u, v)) = v^3\sqrt{u}.$$

Thus, the integral has the value

$$
\begin{aligned}
\iint_R f(x,y)\,dA &= \iint_D f(x(u,v),y(u,v))|J(u,v)|\,du\,dv \\
&= \int_0^1 \int_0^v v^3\sqrt{u}\,|-1|\,du\,dv, \\
&= \int_0^1 v^3 \left[\frac{2}{3}u^{3/2}\right]_0^v dv, \\
&= \frac{2}{3}\int_0^1 v^3 \cdot v^{3/2}\,dv, \\
&= \frac{2}{3}\int_0^1 v^{9/2}\,dv, \\
&= \frac{2}{3}\left[\frac{2}{11}v^{11/2}\right]_0^1, \\
&= \boxed{\frac{4}{33}}
\end{aligned}
$$