Math 320 – Linear Algebra – David Dumas – Fall 2018

Exam 1 Solutions

(1) Let V be a vector space over a field $\mathbb{F}$. Let $w,x,y,z \in V$ and $a \in \mathbb{F}$. Show directly from the vector space axioms that if

$$((aw+x)+ay)+z = \vec{0}$$

then

$$a(w+y) = -(x+z).$$

Justify each step in your proof using one of the vector space axioms. You are not permitted to use any theorems in your solution.

Solution: We are given

$$((aw+x)+ay)+z = \vec{0}.$$

We transform the left hand side by applying several of the axioms:

$$\begin{aligned} (aw+(x+ay))+z &= \vec{0} && \text{By VS2} \\ (aw+(ay+x))+z &= \vec{0} && \text{By VS1 applied to } x+ay \\ ((aw+ay)+x)+z &= \vec{0} && \text{By VS2} \\ (aw+ay)+(x+z) &= \vec{0} && \text{By VS2} \end{aligned}$$

Now, by VS4 there exists an element $-(x+z)$ so that $(x+z)+(-(x+z)) = \vec{0}$. Since the two sides of the last equation above are equal, their sums with $-(x+z)$ are also equal, i.e.

$$((aw+ay)+(x+z))+(-(x+z)) = \vec{0}+(-(x+z))$$

In what follows we refer to the equation above as Equation *.

We consider the two sides of Equation * in turn. First, for the right hand side we have

$$\begin{aligned} \vec{0}+(-(x+z)) &= (-(x+z))+\vec{0} && \text{By VS1} \\ &= -(x+z) && \text{By VS3} \end{aligned}$$

For the left hand side of Equation * we have

$$\begin{aligned} ((aw+ay)+(x+z))+-(x+z) &= (aw+ay)+((x+z)+(-(x+z))) && \text{By VS1} \\ &= (aw+ay)+\vec{0} && \text{By definition of } -(x+z) \\ &= aw+ay && \text{By VS3} \\ &= a(w+y) && \text{By VS7} \end{aligned}$$

Thus we have reduced Equation * to

$$a(w+y) = -(x+z)$$

as required. □

(2) Let $S = \{(1,1,0),(0,1,1),(1,0,1)\}$, a subset of $(\mathbb{Z}_2)^3$. Consider $(\mathbb{Z}_2)^3$ as a vector space over $\mathbb{Z}_2$.
   (a) Is S linearly independent?
   (b) Does S generate $(\mathbb{Z}_2)^3$?
   (c) Is S a basis of $(\mathbb{Z}_2)^3$?
   (d) What is the dimension of $\text{span}(S)$?

Solution:
(a) No. Because $1(1,1,0)+1(0,1,1)+1(1,0,1) = (0,0,0) = \vec{0}$ is a linear combination with not all coefficients zero, the set S is linearly dependent.
(b) No. In fact, we can show that $(1,0,0)$ is not in the span of S, and thus $\text{span}(S) \neq (\mathbb{Z}_2)^3$.
Suppose for contradiction that $a(1,1,0)+b(0,1,1)+c(1,0,1) = (1,0,0)$. Then we have

$$a+c = 1$$
$$a+b = 0$$
$$b+c = 0$$

In $\mathbb{Z}_2$ we have $1 = -1$, and so $a+b = 0$ implies $a = b$, and similarly $b+c = 0$ implies $b = c$. Thus $a = b = c$, and $a+c = a+a$. In $\mathbb{Z}_2$, the sum of any element with itself is zero, hence $a+c = 0$. This contradicts the first equation above.
This contradiction shows that no such coefficients a,b,c exist, and $(1,0,0)$ is not in the span of S. Thus S does not generate.
(c) No. By definition, a basis must be a generating set, and S is not a generating set.
(d) We claim that $\beta = \{(1,1,0),(0,1,1)\}$ is a basis of $\text{span}(S)$, hence $\dim(\text{span}(S)) = 2$.
First, β is linearly independent: If $a(1,1,0)+b(0,1,1) = \vec{0}$ then considering first and last entries gives $a = 0$ and $b = 0$.
Next, we show β generates $\text{span}(S)$. Since $\beta \subset S$, $\text{span}(S)$ is a subspace that contains β, hence by Theorem 1.5, $\text{span}(\beta) \subset \text{span}(S)$.
On the other hand, $1(1,1,0)+1(0,1,1) = (1,0,1)$ shows that $(1,0,1) \in \text{span}(S)$. Since $\beta \cup \{(1,0,1)\} = S$, this shows $S \subset \text{span}(\beta)$. By Theorem 1.5 we have $\text{span}(S) \subset \text{span}(\beta)$.
We have shown $\text{span}(\beta) \subset \text{span}(S)$ and $\text{span}(S) \subset \text{span}(\beta)$, and hence $\text{span}(\beta) = \text{span}(S)$. That is, β generates $\text{span}(S)$.
Since we have shown β is linearly independent and that it generates $\text{span}(S)$, we find that β is a basis. $\square$

(3) Let V be a vector space of dimension n over a field $\mathbb{F}$. Suppose that $\{v_1,\dots,v_n\}$ generates V. Prove that $\{v_1,\dots,v_n\}$ is linearly independent.

Solution: First, it is part of the definition of dimension that every basis of V has exactly n elements; however, this is also easily proved using Theorem 1.10: If β,γ are bases, then applying Theorem 1.10 with $G = \beta$, $L = \gamma$ gives $|\gamma| \leq |\beta|$, while applying the same theorem with $G = \gamma$, $L = \beta$ gives $|\gamma| \geq |\beta|$. Thus $|\beta| = |\gamma|$.
By Theorem 1.9, some subset $\beta \subset \{v_1,\dots,v_n\}$ is a basis. But then $|\beta| = n$, so $\beta = \{v_1,\dots,v_n\}$ is a basis. In particular β is linearly independent. $\square$

(4) Let W denote the set of all polynomials $p \in P_4(\mathbb{R})$ that satisfy $p(1) = 0$. Prove that W is a subspace of $P_4(\mathbb{R})$ and determine the dimension of W.

Solution: Recall $\vec{0} \in P_4(\mathbb{R})$ is the constant polynomial that is equal to zero. Thus $\vec{0}(1) = 0$, and $\vec{0} \in W$. Suppose $p, q \in W$. Then

$$(p+q)(1) = p(1) + q(1) = 0 + 0 = 0$$

which shows $p + q \in W$. Suppose $p \in W$ and $c \in \mathbb{R}$. Then

$$(cp)(1) = c\,p(1) = c0 = 0$$

which shows $cp \in W$. By Theorem 1.3, W is a subspace.

We claim $\beta = \{x-1, x^2-1, x^3-1, x^4-1\}$ is a basis of W. Each of these polynomials satisfies $p(1) = 0$, so they are elements of W.

First, β is linearly independent. Suppose for contradiction that

$$b(x-1) + c(x^2-1) + d(x^3-1) + e(x^4-1) = \vec{0}$$

for some $b, c, d, e \in \mathbb{R}$. Then collecting terms of like degree in the left hand side we find

$$-(b+c+d+e) + bx + cx^2 + dx^3 + ex^4 = \vec{0}$$

and so $b = 0$, $c = 0$, $d = 0$, $e = 0$. This shows β is linearly independent.

Next, we claim β generates W. Suppose $p \in W$ and $p = a + bx + cx^2 + dx^3 + ex^4$. Then

$$0 = p(1) = a + b + c + d + e$$

and so $a = -(a+b+c+d)$. Thus

$$\begin{aligned} p &= -(b+c+d+e) + bx + cx^2 + dx^3 + ex^4 \\ &= b(x-1) + c(x^2-1) + d(x^3-1) + e(x^4-1) \end{aligned}$$

and $p \in \operatorname{span}(\beta)$.

Since we have shown β is linearly independent and that it generates W, we have that β is a basis of W. Therefore $\dim(W) = |\beta| = 4$.