Math 320 – Linear Algebra – David Dumas – Fall 2018

Final Exam Problems 1 and 2

Instructions:

- The final exam will have **six** problems, and you will be asked to complete any **four** of them.
- The first two problems from the final exam are shown below. You may work on these problems in preparation for the exam on December 11. Keep in mind that you will not be able to bring any notes to the exam.
- Collaboration (or receiving assistance of any kind) on these problems is prohibited.

(1) Recall that $P_d(\mathbb{F})$ is the notation for the vector space of polynomials in one variable of degree at most d with coefficients in a field $\mathbb{F}$.

(a) Write an explicit formula* for a linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ that has rank 2, and which has $1+2x+3x^2$ in its null space. Prove that the linear transformation you give as an answer has these properties.

(b) Find a basis for the range of the linear transformation you gave as an answer to part (a).

* To clarify what is meant by an *explicit formula* for such a linear transformation, here is an example: The linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(a+bx+cx^2) = (2b-a)+(3b+c)x+(a+b+c)x^2.$$

Of course, this example does not have the properties requested. It is only provided to show what kind of formula is expected.

(2) Recall that the set $M_{2\times 2}(\mathbb{R})$ of 2×2 matrices with real entries is a vector space over $\mathbb{R}$ of dimension 4. The map $\tau : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ given by $\tau(A) = A^T$, i.e. taking a matrix to its transpose, is a linear operator on this vector space.

(a) Compute the characteristic polynomial of τ.

(b) Find all of the eigenvalues of τ, and for each eigenvalue λ find a basis of the corresponding eigenspace E_λ.

(c) Determine whether or not τ is diagonalizable. If it is diagonalizable, give a basis of $M_{2\times 2}(\mathbb{R})$ that diagonalizes it. If it is not diagonalizable, give a proof of that fact.