Math 550 – David Dumas – Spring 2019

Problem Set 1

Due Wednesday, January 30 in class

Exercises: Work these out, but do not submit them.

(E–) From the textbook: 9.18, 19.4

(E1) Suppose a distribution D is not integrable. Is it possible to define a unique largest subdistribution of D that is integrable? Either describe how to find it, or show that no such thing exists.

(E2) Suppose that the distribution D of rank k on $\mathbb{R}^n$ is defined by taking the span of the vector fields $f_i \frac{\partial}{\partial x_i}$ for $i = 1 \dots k$, where f_i are smooth, nowhere-zero functions. Without using the Frobenius theorem, can you see whether D is integrable? Involutive?

(E3) Suppose $f : M \to N$ is a smooth submersion. Then the fibers $f^{-1}(p)$ are embedded submanifolds of M, all of the same dimension. Is there a distribution D on M for which these fibers are integral submanifolds? (If so, how can you describe D in terms of f?)

Problems: Complete and submit <u>four</u> of these.

(–) Lee, problem 19.4

(–) Lee, problem 19.6

(P1) Let ω be a differential 1-form on a smooth manifold M of dimension n, and suppose that for each $p \in M$ we have $\omega_p \neq 0$. For each $p \in M$ let D_p denote the kernel of the linear map $\omega_p : T_pM \to \mathbb{R}$.
 (a) Show that D is a smooth distribution of rank $n-1$.
 (b) Suppose that ω is closed, i.e. $d\omega = 0$. Show that D is integrable.
 (c) Is $d\omega = 0$ necessary for D to be integrable? Give a proof or a counterexample.

Note: In lecture, we skipped the material on distributions and differential forms on pp.493-496 of Lee, and the answer to this exercise follows from the more general statements proved there. However, do not just cite those general results. Either work this out on your own, or show that you've read and understand that part of the textbook well enough to give a self-contained proof of the special case I'm asking about here.

(P2) Let $\mathrm{GL}_2\mathbb{R}$ denote the Lie group consisting of 2×2 invertible real matrices. The Lie algebra $\mathfrak{gl}_2\mathbb{R}$ of $\mathrm{GL}_2\mathbb{R}$ consists of all 2×2 real matrices. Let $E \subset \mathfrak{gl}_2\mathbb{R}$ be the subspace spanned by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(a) Let $\mathfrak{h}$ denote the smallest Lie subalgebra of $\mathfrak{gl}_2\mathbb{R}$ that contains E. Give a basis for $\mathfrak{h}$ (and in particular determine its dimension).

(b) Find a connected Lie subgroup of $\mathfrak{gl}_2\mathbb{R}$ with Lie algebra $\mathfrak{h}$ in the systematic way that follows from the Frobenius theorem: First find a frame for the distribution corresponding to $\mathfrak{h}$. Then, correct this to a Lie-commuting frame. Finally, use the flows of the commuting frame to parameterize an integral submanifold containing the identity of $\mathfrak{gl}_2\mathbb{R}$.

(Review the material in Chapter 7 and 8 on Lie groups and Lie algebras, and in Chapter 9 on flows, as needed.)