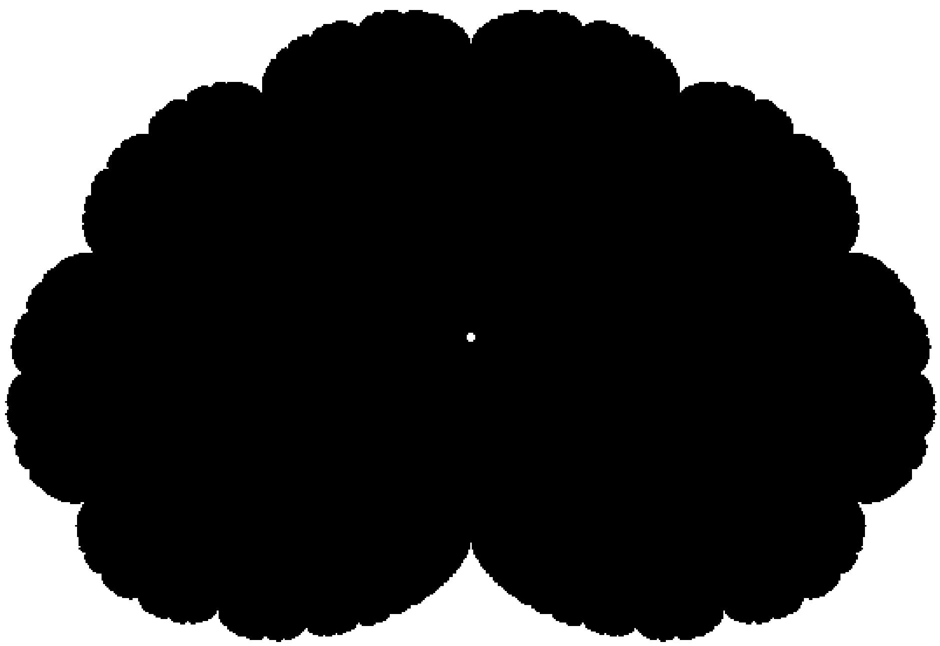
Slicing, Skinning, and Grafting



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(Joint work with Richard Kent)

– Overview –

1. Skinning maps are never constant

2. Bers slices are never algebraic

3. Complex projective structures

4. Fuchsian centers

– Geometrization –

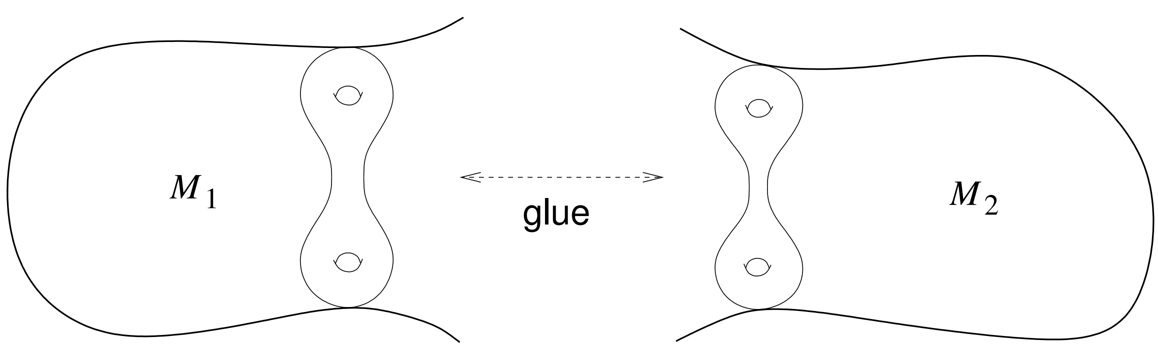
Geometrization Conjecture (Thurston): Compact 3-manifolds can be cut along spheres and tori into geometric pieces.

Thurston proved this for Haken manifolds (around 1980) by showing that a compact atoroidal Haken manifold is hyperbolic.

(Perelman has announced a proof of the complete conjecture.)

The proof for Haken manifolds is divided into two cases: fibered and non-fibered. The latter is an inductive argument using a gluing construction.

Example: Closed manifold N obtained from a disconnected M by gluing components along a surface of genus $g \geq 2$. Given a (complete, infinite volume) hyperbolic metric on M°, want to deform so that the metric is compatible with gluing.



– Skinning Maps –

Thurston turned the gluing problem into a fixed-point problem for a map of Teichmüller space.

Let M be a compact 3-manifold with incompressible boundary, $\chi(\partial M) < 0$ (and for now, no tori), such that M° has a hyperbolic structure.

An extension of Mostow rigidity gives

$$\mathcal{GF}(M) \simeq \mathcal{T}(\partial M)$$

where

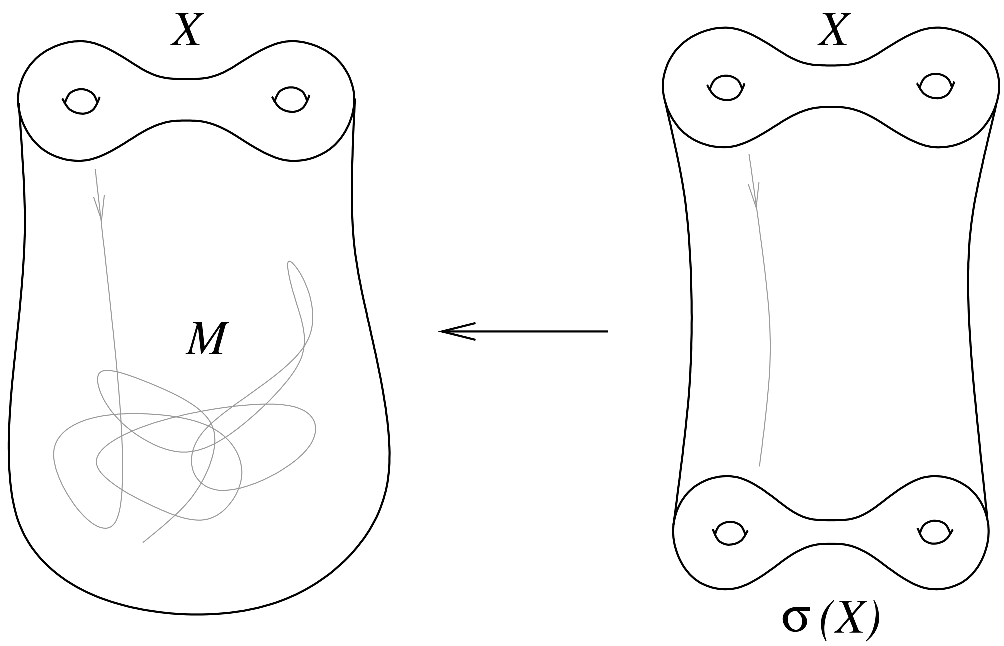
- $\mathcal{GF}(M)$ is the space of geometrically finite hyperbolic structures on M° without cusps
- $\mathcal{T}(\partial M)$ is the Teichmüller space of conformal structures on the boundary.

[Ahlfors, Bers, Kra, Marden, Maskit, Sullivan]

The map $\mathcal{GF}(M) \to \mathcal{T}(\partial M)$ takes a hyperbolic structure to the induced conformal structure on the boundary at infinity.

Suppose that $\partial M = S$ is connected. The cover of M° corresponding to $\pi_1 S$ is diffeomorphic to $S \times \mathbb{R}$. Lifting hyperbolic structures gives

$$\mathcal{GF}(M) \longrightarrow \mathcal{GF}(S \times \mathbb{R}).$$



In terms of the Teichmüller space parameterization, this map is

$$\mathcal{T}(S) \longrightarrow \mathcal{T}(S) \times \mathcal{T}(\overline{S})$$
$$X \longmapsto (X, \sigma(X))$$

This defines $\sigma : \mathcal{T}(S) \to \mathcal{T}(\overline{S})$, the skinning map of M.

For disconnected boundary, one obtains a map for each boundary component, and $\sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M})$ is the product of these.

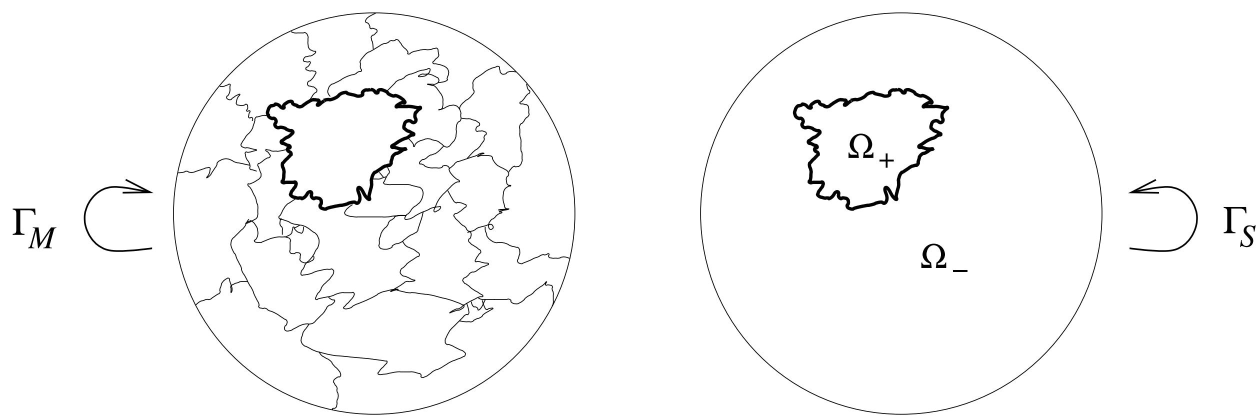
In terms of Kleinian groups: a hyperbolic structure on M° determines

$$\rho : \pi_1 M \longrightarrow \mathsf{PSL}_2(\mathbb{C})$$

an injective map with discrete image Γ_M. The restriction to the boundary is

$$\rho|_{\pi_1 S} : \pi_1 S \longrightarrow \mathsf{PSL}_2(\mathbb{C})$$

whose image is a quasifuchsian group Γ_S.



The limit set of Γ_S is a Jordan curve dividing $\mathbb{CP}^1$ into two domains of discontinuity, $\Omega_\pm$. Thus there are two quotient Riemann surfaces, $\Omega_+/\Gamma = X$ and $\Omega_-/\Gamma = Y$.

Bers: The pair (X, Y) determines Γ_S up to conjugacy, so we write $\Gamma = Q(X, Y)$.

If the hyperbolic structure on M° has conformal boundary $X \in \mathcal{T}(S)$, then the associated quasifuchsian group is $Q(X, \sigma(X))$.

Bounded Image Theorem (Thurston): If M is acylindrical, then $\sigma : \mathcal{T}(\partial M) \to \mathcal{T}(\overline{\partial M})$ has bounded image (i.e. closure of image is compact).

This gives a (partial) solution to the gluing problem: The gluing map induces $\tau : \mathcal{T}(\overline{\partial M}) \to \mathcal{T}(\partial M)$, and a fixed point of $(\tau \circ \sigma)$ is a hyperbolic structure compatible with gluing.

Since $(\tau \circ \sigma)$ is a holomorphic weak contraction with bounded image, iteration converges to a fixed point.

Something else must be done when M has essential cylinders.

(McMullen: Analytic proof that if M is acylindrical, the map σ is uniformly contracting. If cylindrical, iteration converges iff glued manifold is atoroidal.)

Thm 1: Skinning maps are never constant.
That is, let M be a compact 3-manifold with incompressible boundary, $\chi(\partial M) < 0$, M° hyperbolic with no accidental parabolics. Then the skinning map of M is not constant.

[Hypothesis about accidental parabolics simply excludes cylinders joining non-torus and torus boundary components, so if ∂M has no tori it is satisfied.]

– Bers Slices –

The $\mathrm{SL}_2(\mathbb{C})$ character variety $\mathcal{X}(M)$ of a manifold M is the space of representations of $\pi_1 M$ into $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy, i.e.

$$\mathcal{X}(M) = \mathrm{Hom}(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) /\!\!/ \mathrm{SL}_2(\mathbb{C}).$$

Culler-Shalen: The space $\mathcal{X}(M)$ can be realized as an affine $\mathbb{C}$-algebraic variety embedded in $\mathbb{C}^N$ using trace functions.

Choose a finite generating set for $\pi_1(M)$, and let $\mathcal{I} = \{w_1, \ldots, w_N\}$ denote the set of non-repeating words in the generators. Then $\mathcal{X}(M)$ is the image of the map

$$\begin{aligned} \mathrm{Hom}(\pi_1 M, \mathrm{SL}_2(\mathbb{C})) &\longrightarrow \mathbb{C}^N \\ \rho &\longmapsto (\mathrm{tr}\rho(w_i))_{i=1\ldots N} \end{aligned}$$

For a surface S (of genus $g \geq 2$), the variety $\mathcal{X}(S)$ is irreducible and contains the quasifuchsian space

$$\mathcal{QF}(S) = \mathcal{GF}(S \times \mathbb{R}) \simeq \mathcal{T}(S) \times \mathcal{T}(\overline{S})$$

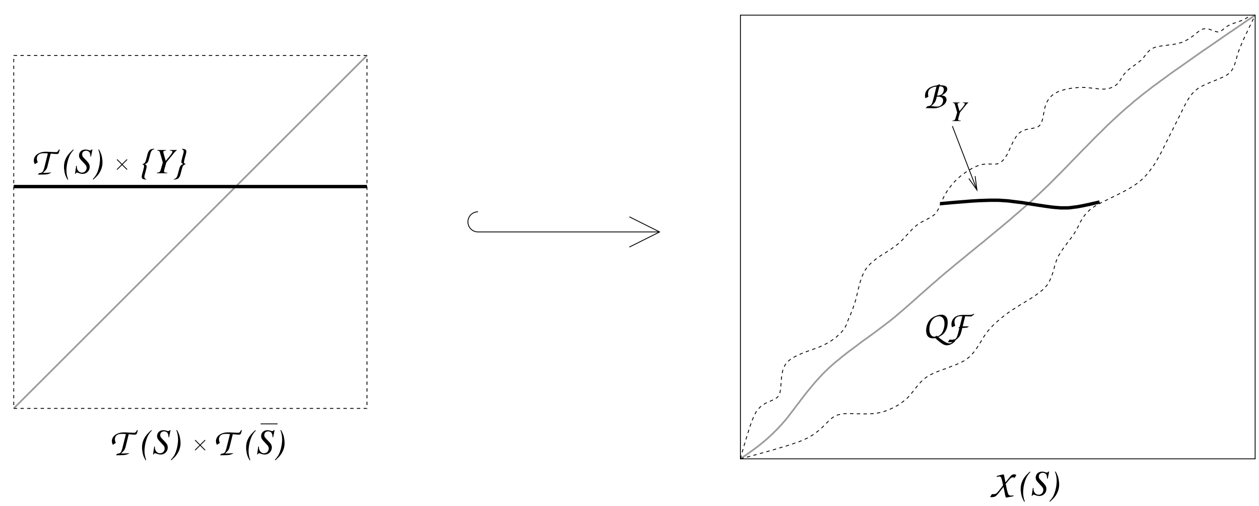
as an open subset of its smooth locus. In particular $\dim \mathcal{X}(S) = 6g - 6$.

(Actually, $\mathcal{X}(S)$ contains 4^g copies of $\mathcal{QF}(S)$ corresponding to different lifts from $\mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$; fix one of them.)

For any $Y \in \mathcal{T}(\overline{S})$, the Bers slice $\mathcal{B}_Y$ is the set

$$\mathcal{B}_Y = \mathcal{T}(S) \times \{Y\} \subset \mathcal{QF}(S) \subset \mathcal{X}(S).$$

Each Bers slice is a holomorphic embedding of Teichmüller space into $\mathcal{X}(S)$, and $\mathcal{QF}(S)$ is the union of these slices.



While each Bers slice $\mathcal{B}_Y$ is bounded (has compact closure) in $\mathcal{X}(S)$, the quasifuchsian space itself is not bounded.

In fact, the diagonal $\{Q(X, \overline{X})\} \subset \mathcal{T}(S) \times \mathcal{T}(\overline{S})$ corresponds to the Fuchsian space $\mathcal{F}(S) \subset \mathcal{QF}(S)$, a properly (but not holomorphically) embedded copy of Teichmüller space.

It would be difficult to directly determine whether a quasifuchsian representation ρ (specified by a set of traces) belongs to a given Bers slice.

One would need to determine the conformal structure on the quotient of the domain of discontinuity of $\rho(\pi_1 S)$, e.g. by uniformization.

(Conversely, it is hard to explicitly determine the effect of quasiconformal conjugation on an element of a Kleinian group.)

Intuitively, it seems that the $(3g-3)$-dimensional subset $\mathcal{B}_Y$ is cut out of $\mathcal{X}(S)$ by transcendental (rather than algebraic) constraints.

Thm 2: Bers slices are never algebraic.
That is, let $\mathcal{V} \subset \mathcal{X}(S)$ be a complex algebraic subvariety of dimension $3g-3$. Then $\mathcal{B}_Y$ is not contained in $\mathcal{V}$.

Equivalently, the Zariski closure of $\mathcal{B}_Y$ has dimension greater than $3g-3$.

Before discussing the proof, we show that Thm 1 (skinning maps are never constant) follows from Thm 2.

– Skinning and Bers Slices –

As before let M be a compact manifold with incompressible boundary, $\chi(\partial M) < 0$, and M° hyperbolizable with no accidental parabolics.

The set of hyperbolic structures $\mathcal{GF}(M)$ is a subset of $\mathcal{X}(M)$ (after choosing a lift from $\mathrm{PSL}_2(\mathbb{C})$) which lies in the smooth locus. Let $\mathcal{X}_0(M)$ be the irreducible component containing $\mathcal{GF}(M)$, so $\dim \mathcal{X}_0(M) = \dim \mathcal{T}(\partial M)$.

Suppose that $S = \partial M$ is connected, so $\dim \mathcal{X}_0(M) = 3g - 3$. The inclusion $\pi_1 S \hookrightarrow \pi_1 M$ induces a regular map of character varieties $r : \mathcal{X}_0(M) \to \mathcal{X}(S)$, which is compatible with the lifting of hyperbolic structures from M° to $S \times \mathbb{R}$:

$$\begin{array}{ccc} \mathcal{X}_0(M) & \xrightarrow{\ r\ } & \mathcal{X}(S) \\ \uparrow & & \uparrow \\ \mathcal{GF}(M) & \longrightarrow & \mathcal{GF}(S \times \mathbb{R}) \end{array}$$

The image $r(\mathcal{X}_0(M))$ is an irreducible algebraic subvariety of $\mathcal{X}(S)$ of dimension $3g - 3$, and it contains all quasifuchsian representations of the form $Q(X, \sigma(X))$.

Thus if the skinning map were constant, say $\sigma(\mathcal{T}(S)) = \{Y\}$, then $r(\mathcal{X}_0(M))$ would contain the Bers slice $\mathcal{B}_Y$, contradicting Thm 2. Thus σ is not constant (∂M connected).

If ∂M is disconnected, but contains no tori, then embed M into a hyperbolizable manifold N with a single incompressible boundary component $S = \partial N \subset \partial M$. (e.g. cap off the other boundary components by gluing them to acylindrical manifolds with connected incompressible boundary.)

Then the skinning map of N (which is not constant) factors through that of M:

$$\begin{array}{ccc} \mathcal{T}(\partial M) & \xrightarrow{\sigma_M} & \mathcal{T}(\overline{\partial M}) \\ \uparrow & & \downarrow \\ \mathcal{T}(\partial N) & \xrightarrow{\sigma_N} & \mathcal{T}(\overline{\partial N}) \end{array}$$

The vertical map at left is $\mathcal{GF}(N) \to \mathcal{GF}(M)$ induced by the embedding $M \hookrightarrow N$, while $\mathcal{T}(\overline{\partial M}) \to \mathcal{T}(\overline{\partial N})$ is the projection to one factor. Thus σ_M is not constant.

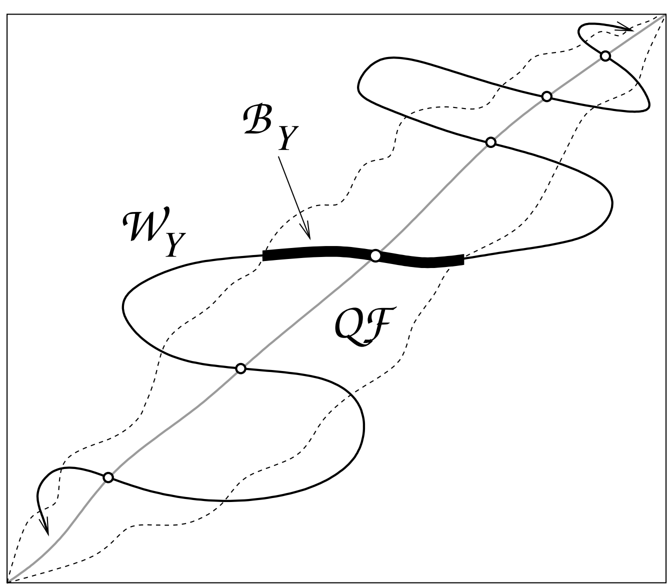
Finally, if ∂M contains tori, the same argument can be applied to the subvariety of $\mathcal{X}(M)$ in which each peripheral $\mathbb{Z} \oplus \mathbb{Z}$ has parabolic image.

Now we turn to the proof of Thm 2 (Bers slices are never algebraic).

We must show that there is no algebraic subvariety $\mathcal{V} \subset \mathcal{X}(S)$ of dimension $3g-3$ that contains a Bers slice. Can assume that $\mathcal{V}$ is irreducible.

There are two steps:

1. The Bers slice $\mathcal{B}_Y$ is contained in a complex *analytic* subvariety $\mathcal{W}_Y \subset \mathcal{X}(S)$ of dimension $3g-3$ (using holonomy of projective structures). Thus this is the only candidate for $\mathcal{V}$.
2. The analytic variety $\mathcal{W}_Y$ has infinitely many isolated real points (the Fuchsian centers), and is therefore not algebraic.



$\mathcal{X}(S)$

– Projective Structures –

Let $Y \in \mathcal{T}(S)$ be a complex structure on the compact surface S.

A (complex) projective structure on Y is an atlas of holomorphic charts with values in $\mathbb{CP}^1$ and Möbius transition functions.

There is a complex-analytic description of the set of all projective structure on Y up to isomorphism:

After lifting to the universal cover $\tilde{Y} \simeq \Delta$, one can adjust projective charts by Möbius transformations to agree on overlaps, giving a globally defined and locally injective developing map $f : \Delta \to \mathbb{CP}^1$.

The Schwarzian derivative is a Möbius-invariant differential operator on meromorphic functions:

$$S(f) = \left[\left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right] dz^2$$

The Schwarzian of the developing map, $S(f)$, is a $\pi_1 S$-invariant quadratic differential $\tilde{\phi}$ on Δ, which descends to a holomorphic quadratic differential ϕ on Y.

This quadratic differential uniquely determines the projective structure, because the equation $S(f) = \tilde{\phi}$ has a unique solution up to composition with Möbius transformations.

In fact, if u_1, u_2 are linearly independent holomorphic solutions of the linear ODE

$$u'' + \frac{1}{2}\tilde{\phi}\, u = 0,$$

then $f = u_1/u_2$ satisfies $S(f) = \tilde{\phi}$. Changing the basis u_1, u_2 for the solution space is equivalent to composing f with a Möbius transformation.

Thus the space of projective structures on Y is identified with $Q(Y) \simeq \mathbb{C}^{3g-3}$, the vector space of holomorphic quadratic differentials.

(Note: This construction depends on the identification $\tilde{Y} \simeq \Delta$. The uniformization-independent version is that the Schwarzian measures the difference between projective structures, making the space of projective structures on Y into an affine space.)

– Holonomy –

Given $\phi \in Q(Y)$, let $f : \tilde{Y} \to \mathbb{CP}^1$ be the developing map of the associated projective structure. For each $\gamma \in \pi_1 S$, the germs of f at $z \in \tilde{Y}$ and at $\gamma(z)$ differ by composition with a Möbius transformation A_γ.

The map $\gamma \mapsto A_\gamma$ defines the holonomy representation of the projective structure, $\mathrm{hol}(\phi) : \pi_1 S \to \mathrm{PSL}_2(\mathbb{C})$. This representation lifts to $\mathrm{SL}_2(\mathbb{C})$, and allowing ϕ to vary gives the holonomy map

$$\mathrm{hol} : Q(Y) \longrightarrow \mathcal{X}(S).$$

This map is a proper holomorphic embedding. (Properness was proved by Gallo-Kapovich-Marden. Earlier, Tanigawa showed that the map is proper into the subset of $\mathcal{X}(S)$ consisting of irreducible characters.)

Alternatively, $\mathrm{hol}(\phi)$ is the (projectivization of the) holonomy of the second-order ODE used to solve the Schwarzian equation $S(f) = \tilde{\phi}$. (The invariant formulation is that ϕ determines a flat holomorphic connection on a rank 2 vector bundle over Y.)

Thus the image $\mathcal{W}_Y = \mathrm{hol}(Q(Y))$ is a complex analytic subvariety of $\mathcal{X}(S)$ of dimension $3g - 3$.

Quasifuchsian groups of the form $\Gamma = Q(X,Y)$ give examples of projective structures on Y: The charts are local inverses of the universal covering $\Omega_- \to Y$, where Ω_- is one of the domains of discontinuity of Γ. (Equivalently, the Riemann map $\Delta \to \Omega_-$ is the developing map.)

By construction, the holonomy of the projective structure obtained from $Q(X,Y)$ in this way is simply $Q(X,Y)$.

(Mapping X to the Schwarzian of the projective structure $Q(X,Y)$ gives the Bers embedding $\mathcal{T}(S) \hookrightarrow Q(Y)$. This was the first holomorphic embedding of Teichmüller space into a complex vector space.)

As a result, the analytic variety $\mathcal{W}_Y \subset \mathcal{X}(S)$ contains the Bers slice $\mathcal{B}_Y$.

In fact, the ball of radius 1/2 in $Q(Y)$ maps into $\mathcal{B}_Y$, and the ball of radius 3/2 covers $\mathcal{B}_Y$. Here we use the hyperbolic L^∞ norm on $Q(Y)$. [Nehari, Krauss]

– Uniqueness –

Suppose there exists an irreducible algebraic subvariety $\mathcal{V} \subset \mathcal{X}(S)$ of dimension $3g-3$ containing $\mathcal{B}_Y$.

Since hol : $Q(Y) \to \mathcal{X}(S)$ is holomorphic, and an open subset of $Q(Y)$ maps to $\mathcal{B}_Y \subset \mathcal{V}$, we have $\mathcal{W}_Y = \text{hol}(Q(Y)) \subset \mathcal{V}$. (The pullback by hol of the functions defining $\mathcal{V}$ are holomorphic on $Q(Y)$ and vanish on an open set.)

Let $\mathcal{V}^*$ denote the set of smooth points of $\mathcal{V}$, a connected complex manifold of dimension $3g-3$.

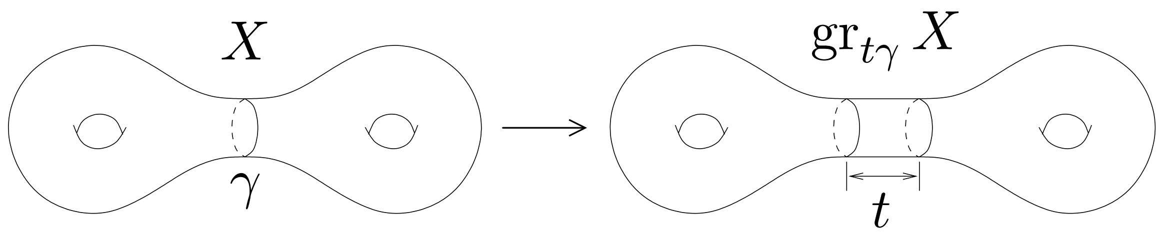
Since hol is proper, $\mathcal{W}_Y \cap \mathcal{V}^*$ is a properly embedded submanifold of $\mathcal{V}^*$ of the same dimension. Thus $\mathcal{W}_Y \cap \mathcal{V}^* = \mathcal{V}^*$.

But $\mathcal{V}^*$ is dense in $\mathcal{V}$, and $\mathcal{W}_Y$ is closed, so $\mathcal{V} = \mathcal{W}_Y$.

So to complete the proof of Thm 2, we need only show that $\mathcal{W}_Y$ is not an algebraic subvariety.

– Grafting –

Start with X, a closed hyperbolic surface, and γ, a simple closed hyperbolic geodesic. Cut X along γ and insert a Euclidean cylinder of length t.



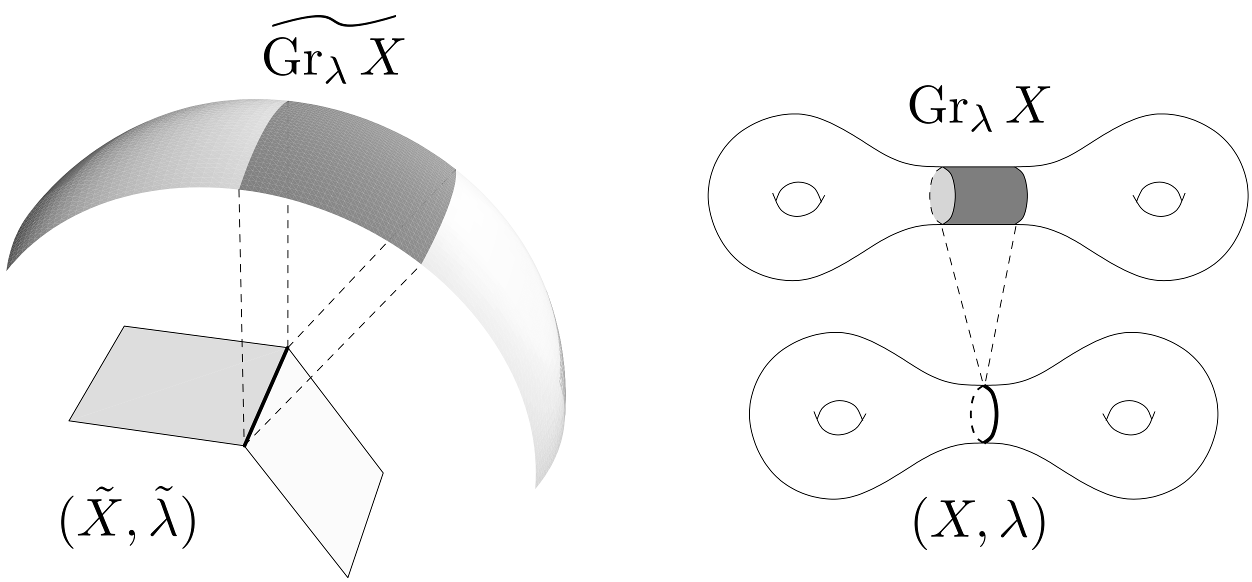
The result is a surface $\mathrm{gr}_{t\gamma} X$, the grafting of X by $t\gamma$, which has a $C^{1,1}$ Riemannian metric (part hyperbolic, part Euclidean) and a well-defined conformal structure.

Thus we can consider $\mathrm{gr}_{t\gamma} X$ as a Riemann surface. Note that it is not conformally equivalent to X.

By inserting several cylinders, grafting extends naturally to a finite disjoint union of weighted geodesics $\lambda = \sum t_i \gamma_i$ (and more generally, to measured laminations).

A construction of Thurston equips a grafted surface $\mathrm{gr}_\lambda X$ with a canonical projective structure, denoted $\mathrm{Gr}_\lambda X$.

Essentially, the projective charts of $\mathrm{Gr}_\lambda X$ are given by the Gauss map of a locally convex pleated plane in $\mathbb{H}^3$ obtained by bending $\tilde{X} \simeq \mathbb{H}^2$ along lifts of λ.



Equivalently, there are natural projective structures on the hyperbolic surface X (from its uniformization by $\mathbb{H}$) and on the cylinders (from the covering by a sector in $\mathbb{C}^*$ with deck group $z \mapsto \lambda z$), and these fit together to form $\mathrm{Gr}_\lambda X$.

When λ is 2π-integral, i.e. $\lambda = \sum 2\pi n_i \gamma_i$ with $n_i \in \mathbb{Z}^+$, the holonomy of $\mathrm{Gr}_\lambda X$ is the Fuchsian group $Q(\overline{X}, X)$. (Because rotation by $2\pi n$ is the identity.)

This gives infinitely many projective structures with the same Fuchsian holonomy, parameterized by $\mathcal{ML}_{2\pi\mathbb{Z}}$, the set of 2π-integral measured laminations. But these are projective structures on different Riemann surfaces (of the form $\mathrm{gr}_\lambda X$).

However, the map $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ is a diffeomorphism when λ is 2π-integral. [Tanigawa; later Scannell-Wolf extended this to all laminations]

So given Y and a 2π-integral lamination λ there is a projective structure on Y (equivalently, an element of $Q(Y)$) with Fuchsian holonomy obtained by λ-grafting. This gives a map $\Psi : \mathcal{ML}_{2\pi\mathbb{Z}} \to Q(Y)$, i.e.

$$\Psi(\lambda) = \mathrm{Gr}_\lambda \left(\mathrm{gr}_\lambda^{-1}(Y) \right).$$

Because λ can be recovered from the topology of the associated developing map, the map Ψ is injective. Extremal length estimates show that its image is discrete.

(In fact Ψ extends to a homeomorphism $\mathcal{ML} \to Q(Y)$.)

– Fuchsian centers –

Projective structures of the form $\Psi(\lambda) \in Q(Y)$, $\lambda \in \mathcal{ML}_{2\pi\mathbb{Z}}$ are called Fuchsian centers.

(Each is a distinguished center point for a surrounding "island" of projective structures with quasifuchsian holonomy.)

The Fuchsian centers are the only projective structures on Y with Fuchsian holonomy. That is, every structure with Fuchsian holonomy arises from integral grafting. [Goldman]

Now we use the Fuchsian centers to study the holonomy set $\mathcal{W}_Y = \mathrm{hol}(Q(Y))$:

Within $\mathcal{X}(S)$ there is the subset $\mathcal{X}_{\mathbb{R}}(S)$ of representations with real characters (i.e. the trace of the image of every element of $\pi_1 S$ is real), which is a real algebraic variety.

Each representation in $\mathcal{X}_{\mathbb{R}}(S)$ is conjugate into either $\mathrm{SL}_2(\mathbb{R})$ (e.g. Fuchsian representations) or $\mathrm{SU}(2)$. [Bass-Morgan-Shalen]

Furthermore the Fuchsian representations $\mathcal{F}(S)$ form an irreducible and connected component of $\mathcal{X}_{\mathbb{R}}(S)$.

If $\mathcal{W}_Y$ were a complex algebraic variety, then its intersection $\mathcal{W}_Y \cap \mathcal{X}_\mathbb{R}$ would be a real algebraic variety, and in particular it would have finitely many connected components. [Whitney]

But each Fuchsian center gives a point in $\mathrm{hol}(\Psi(\lambda)) \in (\mathcal{W}_Y \cap \mathcal{X}_\mathbb{R}(S))$, which is isolated among the (discrete) subset of $\mathcal{W}_Y$ with Fuchsian holonomy.

Since Fuchsian representations form a connected component of $\mathcal{X}_\mathbb{R}(S)$, each Fuchsian center is actually an isolated point of $(\mathcal{W}_Y \cap \mathcal{X}_\mathbb{R}(S))$, and this set has infinitely many connected components.

Thus $\mathcal{W}_Y$ is not algebraic, and Thm 2 follows.

(Goldman's classification of Fuchsian holonomy is not strictly necessary, since all we used was that Fuchsian centers exist, and the set with Fuchsian holonomy is discrete. The latter was proved earlier by Faltings, using a differential calculation.)

– Discussion & Questions –

Zariski closure

A Bers slice is not algebraic, so its Zariski closure has strictly larger dimension. However the proof of Thm 2 gives no insight into the structure of the Zariski closure (except that it contains $\mathcal{W}_Y$ and the infinite set of Fuchsian centers).

The character variety $\mathcal{X}(S)$ is irreducible, so it seems natural to ask:

- Are Bers slices Zariski dense in $\mathcal{X}(S)$?

A better understanding of the analytic subvariety $\mathcal{W}_Y$ and its parameterization by $Q(Y)$ seems like a good place to start, e.g.

- How does the geometry of Y determine the set of $\mathrm{SL}_2(\mathbb{C})$ characters $\mathcal{W}_Y$?

- What is the limiting behavior of $\mathcal{W}_Y$ in $\mathcal{X}(S)$? (Say, in the Morgan-Shalen compactification?) [Gallo-Kapovich-Marden]

Generalized Bers slices

Bers slices admit various limit constructions to produce other deformation spaces of Kleinian surface groups parameterized by $\mathcal{T}(S)$.

For example, one can pinch a collection of curves to obtain a generalized Bers slice $\mathcal{B}_{\{\gamma_i\},Y_0}$ consisting of geometrically finite groups with a fixed collection of accidental parabolics $\{\gamma_i\}$ and punctured conformal structure Y_0 on one end.

These geometrically finite generalized Bers slices are not Zariski dense; each pinched curve gives an algebraic condition on the associated characters ($\text{tr}(\gamma) = \pm 2$).

In fact, when a maximal collection of $3g-3$ curves is pinched, the result is algebraic. (e.g. the Maskit slice for punctured tori)

Given an irrational ending lamination ε, one can also consider the family B_ε of Kleinian surface groups with one geometrically infinite end with ending lamination ε and one geometrically finite end without cusps.

(Alternatively, given one such group Γ_ε, let $\mathcal{B}_\varepsilon$ be its quasiconformal deformation space.)

Combinations of these constructions are possible (for example, a fixed end with some accidental parabolics and some degenerate ends of the pared manifold). Given Thm 2, it is natural to ask:

- Which generalized Bers slices are algebraic?
- What are their Zariski closures?

If ε is fixed by a pseudo-anosov mapping class, then the stable manifold is a natural analog of $\mathcal{W}_Y$. But for general ε, it is not even clear if one can find a complex-analytic map $\mathbb{C}^{3g-3} \to \mathcal{X}(S)$ that surjects $\mathcal{B}_\varepsilon$.

Surfaces with punctures

One might also expect Thm 2 to hold for Bers slices of finitely punctured compact Riemann surfaces, though properness of the monodromy is not known in this case.

(Goldman's classification of Fuchsian structures and Tanigawa's result on integral grafting are also only proved for compact surfaces. Faltings' transversality includes the punctured case.)

Moduli space of holomorphic connections

Gunning described projective structures on Y in terms of an affine space of holomorphic connections on a maximally unstable rank 2 vector bundle.

There is a moduli space $\mathcal{M}_{DR}(Y)$ of bundles over Y with holomorphic connections, and Simpson showed that the holonomy map $\mathcal{M}_{DR}(Y) \to \mathcal{X}(S)$ is complex-analytic but *not* algebraic.

The argument uses the theory of Higgs bundles.

However a non-algebraic map can have algebraic image, and can map a subvariety to a subvariety.

- Can Higgs bundle or vector bundle techniques be used to show that $\mathcal{W}_Y$ is not algebraic? (that it is Zariski dense?)

Local versus global

The proof of Thm 2 shows that the Bers slice is not algebraic by exhibiting an infinite set of isolated real points of $\mathcal{W}_Y$, but only one of these (with $\lambda = 0$) lies in $\mathcal{B}_Y$.

- Within $\mathcal{B}_Y$, is there a local obstruction to the existence of an algebraic variety $\mathcal{V}$ of dimension $3g-3$ containing $\mathcal{B}_Y$?