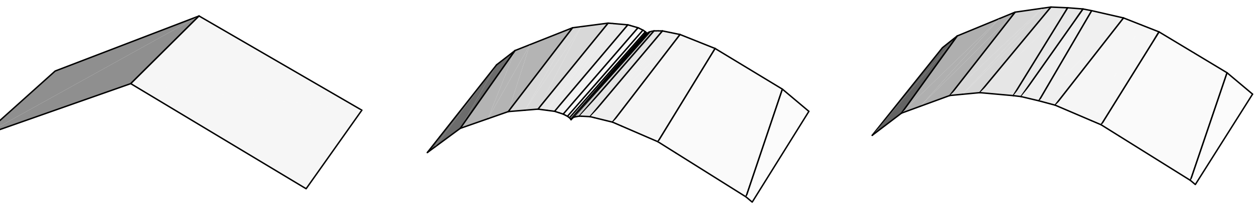
Grafting Coordinates for Teichmüller Space



October 2006

David Dumas (ddumas@math.brown.edu)
http://www.math.brown.edu/~ddumas/

(Joint work with Mike Wolf)

– Grafting –

Start with X, a closed hyperbolic surface, and γ, a simple closed hyperbolic geodesic. Cut X along γ and insert a Euclidean cylinder of length t.

The result is $\mathrm{gr}_{t\gamma} X$, the **grafting** of X along $t\gamma$.

Grafting extends continuously to limits of weighted geodesics, i.e. **measured laminations**. Intuitively, grafting replaces λ with a thickened version that has a Euclidean metric. [Thurston; Kamishima-Tan]

Thus grafting defines a continuous map

$$\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$$

where S is the smooth surface underlying X, $\mathscr{T}(S)$ is Teichmüller space, $\mathscr{ML}(S)$ is the PL-manifold of measured laminations.

Note $\dim \mathscr{T}(S) = \dim \mathscr{ML}(S) = 6g - 6$.

Main Theorem *For each $X \in \mathscr{T}(S)$, grafting X defines a homeomorphism $\mathrm{gr}_\bullet X : \mathscr{ML}(S) \to \mathscr{T}(S)$ (i.e. $\lambda \mapsto \mathrm{gr}_\lambda X$).*

Actually, we show that this map is a tangentiable diffeomorphism. Its (lack of) regularity is a key issue.

This is a natural complement to:

Theorem (Scannell-Wolf) *For each $\lambda \in \mathscr{ML}(S)$, the λ-grafting map $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a diffeomorphism.*

Our proof of the main theorem uses the Scannell-Wolf theorem and a complex-linearity technique of Bonahon.

– Proof Outline: Scannell-Wolf Thm –

- Tanigawa: $\mathsf{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is proper
- Thus it suffices to show that gr_λ is a local diffeomorphism, for then gr_λ is a smooth covering map of a contractible space.
- By the inverse function theorem, it suffices to study the derivative of gr_λ.
- Infinitesimal Sc-W theorem: The derivative

  $$d\,\mathsf{gr}_\lambda : T_X\mathscr{T}(S) \to T_{\mathsf{gr}_\lambda X}\mathscr{T}(S)$$

  is an injective linear map ($\Rightarrow$ isomorphism).
- Proof of infinitesimal theorem is a PDE argument based on the prescribed curvature and geodesic equations, applied to the **Thurston metric** on $\mathsf{gr}_\lambda X$ (i.e. hyperbolic on X, Euclidean on the cylinder).

– Proof Outline: Main Theorem –

- Tanigawa: $\mathrm{gr}_\bullet X : \mathscr{ML}(S) \to \mathscr{T}(S)$ is proper
- As before, it then suffices to show that $\mathrm{gr}_\bullet X$ is a local homeomorphism.
- Bonahon: Grafting is **tangentiable** ($\simeq$ has one-sided derivatives everywhere). So an infinitesimal analysis is possible, can reduce to:
- **Infinitesimal Main Thm** *If a (PL or tang'ble) family λ_t of measured laminations satisfies $\frac{\partial}{\partial t}\Big|_{t=0^+} \mathrm{gr}_{\lambda_t} X = 0$, then $\frac{\partial}{\partial t}\Big|_{t=0^+} \lambda_t = 0$.*

  (The **tangent map** of $\mathrm{gr}_\bullet X$ has no kernel at λ_0.)
- Given a supposed counterexample (λ_t, X) to the infinitesimal main thm, use **shearing** to create a family $X_t \in \mathscr{T}(S)$ such that

$$i\left[\frac{\partial}{\partial t}\Big|_{t=0^+} \mathrm{gr}_{\lambda_0} X_t\right] = \frac{\partial}{\partial t}\Big|_{t=0^+} \mathrm{gr}_{\lambda_t} X = 0.$$

  Thus $\frac{\partial}{\partial t}\Big|_{t=0^+} X_t = 0$. The way X_t is constructed then gives $\frac{\partial}{\partial t}\Big|_{t=0^+} \lambda_t = 0$.
- This relationship between derivatives comes from Bonahon's theory of shear-bend cocycles and the complex duality between shearing and bending.

– Derivatives in $\mathscr{ML}(S)$ and Cocycles –

Let λ_t be a PL family of laminations, $t \in [0, \epsilon)$. (We use PL instead of tangentiable for simplicity.)

Want to make sense of the derivative $\dot{\lambda} = \frac{\partial}{\partial t}\Big|_{t=0^+} \lambda_t$ (following Bonahon, Thurston).

One way is to put λ_t in a **train track chart**. Then the derivatives of the edge weights at $t = 0^+$ give a **signed measure** on the train track.

Let Λ be the **essential support** of λ_t at $t = 0$, i.e.

$$\Lambda = \lim_{t \to 0^+} \Lambda_t \quad \text{where} \quad \Lambda_t = \text{supp}(\lambda_t),$$

using the Hausdorff topology on geodesic laminations. Typically Λ is bigger than the support of λ_0.

The derivative $\dot{\lambda}$ can be interpreted as a **transverse cocycle** (finitely additive signed transverse measure) for Λ.

Can describe this cocycle using the train track derivative, or directly in terms of intersection numbers:

$$i(\dot{\lambda}, \tau) := \lim_{t \to 0+} \frac{i(\lambda_t, \tau) - i(\lambda_0, \tau)}{t}$$

Assume Λ is maximal (complementary regions are ideal triangles) by enlarging it if necessary. Let $\mathcal{H}(\Lambda)$ be the vector space of transverse cocycles on Λ.

$$\mathcal{H}(\Lambda) \simeq \mathbb{R}^{6g-6}$$

So the family λ_t determines a vector $\dot{\lambda} \in \mathcal{H}(\Lambda)$.

Idea to construct X_t: Embed $\mathscr{T}(S)$ in $\mathcal{H}(\Lambda)$, then translate X by $t\dot{\lambda}$ in this embedding to obtain X_t.

− Shearing −

Given a maximal geodesic lamination Λ, Bonahon defines an embedding $\sigma : \mathscr{T}(S) \to \mathcal{H}(\Lambda)$, where $\sigma(X)$ is the **shearing cocycle** of X:

The lift of Λ to $\tilde{X} \simeq \mathbb{H}^2$ is a tiling by ideal triangles (not necessarily locally finite).

The value of $\sigma(X)$ on a transversal τ (lifted to $\mathbb{H}^2$) connecting triangles T_P and T_Q is the **relative shear** of T_P and T_Q.

For example, if T_P and T_Q share an edge, then $i(\sigma(X), \tau)$ is the signed distance between the feet of the altitudes of T_P and T_Q on this edge.

Otherwise, identify the nearest edges of T_P and T_Q using "fans" of geodesics interpolating between the leaves of Λ separating T_P and T_Q.

Thm (Bonahon) *The map* $\sigma : \mathscr{T}(S) \to \mathcal{H}(\Lambda)$ *is a real-analytic embedding; its image is an open convex cone with finitely many faces.*

Thus for all t sufficiently small, the sum $\sigma(X) + t\dot{\lambda}$ is the shearing cocycle of some $X_t \in \mathscr{T}(S)$, and $X_0 = X$. This is a **shearing** or **cataclysm path**.

For example, if $\dot{\lambda}$ is supported on a singled closed geodesic γ, then X_t is obtained by twisting X along γ; if $\dot{\lambda}$ is a positive measure, then X_t is the associated earthquake path.

– Completing the proof –

Finally, we use a remarkable complex linearity property of the derivative of grafting (with respect to the shearing embedding):

Thm (Bonahon) *Let $Y_t = \mathrm{gr}_{\lambda_t} X_t$ where $\dot\lambda \in \mathcal{H}(\Lambda)$ and $\dot\sigma = \frac{\partial}{\partial t}\big|_{t=0^+} \sigma(X_t) \in \mathcal{H}(\Lambda)$. Then $\dot Y = \frac{\partial}{\partial t}\big|_{t=0^+} Y_t$ is a $\mathbb{C}$-linear function of the complex cocycle $(\dot\sigma + i\dot\lambda) \in \mathcal{H}(\Lambda) \otimes \mathbb{C}$.*

Recall that we started with (λ_t, X_0) such that $\mathrm{gr}_{\lambda_t} X_0$ is constant to first order, and then used $\dot\lambda \in \mathcal{H}(\Lambda)$ to construct a shearing path X_t.

Applying the $\mathbb{C}$-linearity theorem to (λ_0, X_t) and (λ_t, X_0) (with associated complex cocycles $\dot\lambda$ and $i\dot\lambda$, resp.) we find:

$$i\left[\frac{\partial}{\partial t}\bigg|_{t=0^+} \mathrm{gr}_{\lambda_0} X_t\right] = \frac{\partial}{\partial t}\bigg|_{t=0^+} \mathrm{gr}_{\lambda_t} X_0 = 0.$$

By Scannell-Wolf, $\frac{\partial}{\partial t}\big|_{t=0^+} X_t = 0$, but in the shearing embedding $\frac{\partial}{\partial t}\big|_{t=0^+} X_t = \dot\lambda$. Thus $\dot\lambda = 0$. $\square$

– Why $\mathbb{C}$-linearity? –

Thurston connected grafting with $\mathbb{CP}^1$ structures on surfaces. The idea is to lift to the universal cover and exploit a natural equivalence:

$$(\text{Grafting } \Delta \subset \mathbb{C} \subset \mathbb{CP}^1) \;\leftrightarrow\; (\text{Bending } \mathbb{H}^2 \subset \mathbb{H}^3)$$

This allows one to understand the derivative of grafting by studying the effect of a bending deformation on the holonomy of a pleated surface (Bonahon; Epstein-Marden).

The ultimate "source" of the complex linearity is: *The hyperbolic isometry with translation s and twist t along a fixed axis is a holomorphic function of $(s + it)$.*

– Applications –

- Comparing geometric and analytic perspectives on $\mathbb{CP}^1$ structures.

  Every $\mathbb{CP}^1$ structure is obtained by **projective grafting**, giving $\mathbb{CP}^1(S) \simeq \mathscr{ML}(S) \times \mathscr{T}(S)$.

  Strata in $\mathbb{CP}^1(S)$ with constant complex structure project homeomorphically to both $\mathscr{ML}(S)$ and $\mathscr{T}(S)$ (by Scannell-Wolf and main thm, respectively).

- Hyperbolic structure on convex hull boundary parameterizes a Bers slice.

  Let M be a quasi-Fuchsian hyperbolic structure on $S \times \mathbb{R}$ with ideal boundary $Y \cup Y'$ and convex core boundary $X \cup X'$. Then M is determined up to isometry by $(X, Y) \in \mathscr{T}(S) \times \mathscr{T}(S)$.

– Applications –

- Grafting coordinates, grafting rays.

  For each $X \in \mathscr{T}(S)$, the map $\lambda \mapsto \text{gr}_\lambda X$ gives "polar coordinates" for $\mathscr{T}(S)$ centered at X.

  Each ray in $\mathscr{ML}(S)$ maps to a **grafting ray** $\{\text{gr}_{t\lambda} X\}_{t \in \mathbb{R}+}$ in $\mathscr{T}(S)$, a properly embedded smooth path starting at X.

  Intuition:

  – For small t, the λ-ray is like i(twist), because the grafting cylinder is nearly geodesic.

  – For large t, it is like a Teichmüller deformation with horizontal foliation λ, because the grafting cylinder nearly fills S.

  Properties of the γ-ray (γ=simple closed geodesic):

  – Tangent vector at $t = 0$ is $\nabla_{\text{WP}}(\ell_\gamma)$ (Wolpert; McMullen).

  – Extremal length of γ is eventually monotone decreasing

  – Hyperbolic length of γ is eventually monotone decreasing