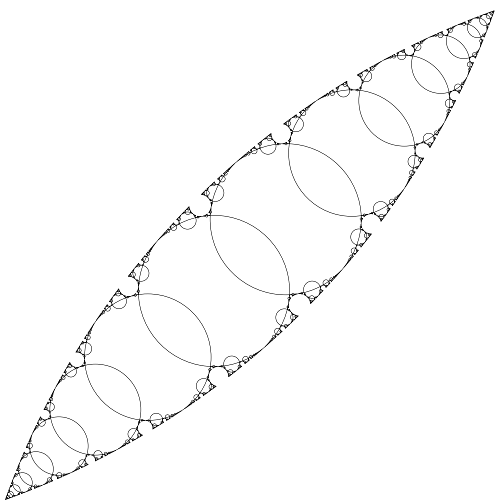
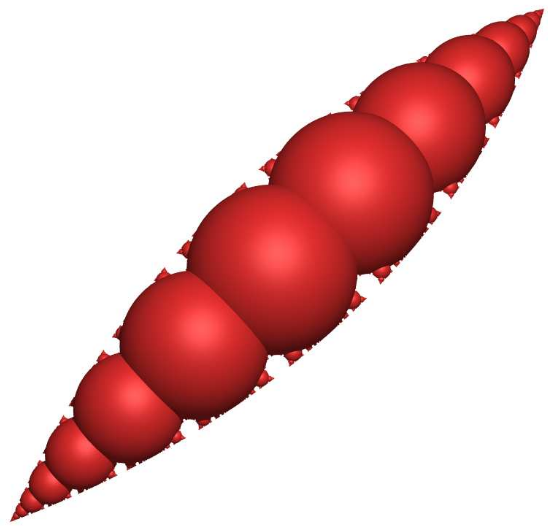
Analysis and Geometry of $\mathbb{CP}^1$ Structures on Surfaces





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(Includes joint work with Mike Wolf)

– Overview –

1. A $\mathbb{CP}^1$ structure on a surface can be studied using

   - **Analysis**
     Schwarzian derivative, univalent functions, harmonic maps, . . .

   - **Geometry**
     Grafting, pleated surfaces in $\mathbb{H}^3$, Kleinian groups, . . .

2. Each perspective leads to a model (coordinate system) for the moduli space $\mathscr{P}(S)$ of marked $\mathbb{CP}^1$ surfaces.

3. **Goal:** Understand the relationship between the two perspectives.

4. Will discuss several results toward the goal, and a qualitative model for $\mathscr{P}(S)$ and its two coordinate systems.

- $\mathbb{CP}^1$ Structures -

Fix a compact smooth surface S of genus $g \geq 2$.

A $\mathbb{CP}^1$ (or Möbius) structure on S is an atlas of charts with values in $\mathbb{CP}^1$ and Möbius transition functions.

Example: boundary of a hyperbolic 3-manifold.

Let $\mathscr{P}(S)$ denote the space of marked $\mathbb{CP}^1$ structures on S.

Underlying a $\mathbb{CP}^1$ structure is a complex structure, since Möbius transformations are holomorphic.

Thus $\mathscr{P}(S)$ has a natural "forgetful" map to the Teichmüller space $\mathscr{T}(S)$.

$$\pi : \mathscr{P}(S) \to \mathscr{T}(S)$$

Let $P(X) = \pi^{-1}(X)$ denote the projective structures with underlying complex structure X. As X varies, $P(X)$ foliate $\mathscr{P}(S)$.

- Schwarzian Derivative -

The Schwarzian derivative is a Möbius-invariant differential operator on meromorphic functions:

$$S(f) = \left[\left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2\right] dz^2$$

The Schwarzian derivatives of the charts of a $\mathbb{CP}^1$ structure on X assemble to a holomorphic quadratic differential $\phi \in Q(X)$.

In fact the Schwarzian defines an isomorphism $Q(X) \simeq P(X)$, and thus $\mathscr{P}(S)$ is identified with the cotangent bundle of $\mathscr{T}(S)$.

$$\mathscr{P}(S) \simeq T^*\mathscr{T}(S)$$

This is the analytic parameterization of $\mathscr{P}(S)$:

A projective structure on S is uniquely determined by its underlying complex structure X and its Schwarzian ϕ.

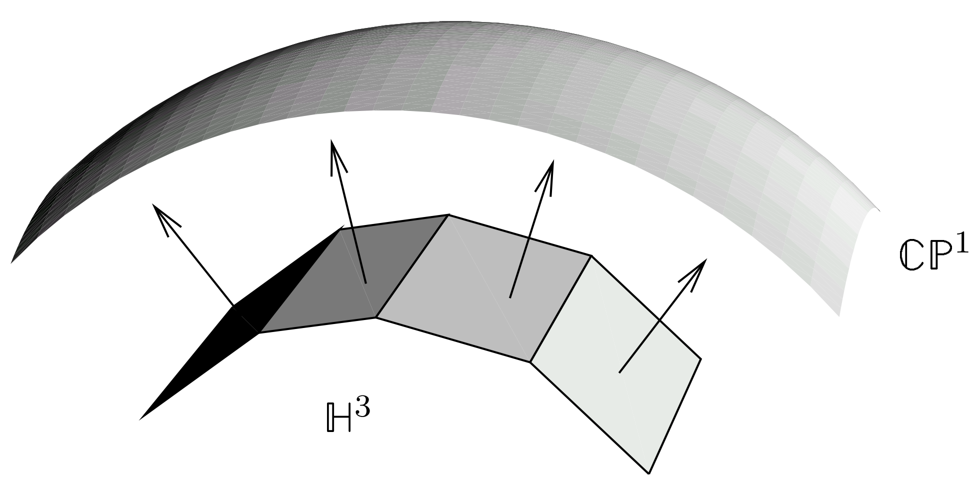
- Convex Hulls and Grafting -

On a $\mathbb{CP}^1$ surface, there is a well-defined notion of a *round disk*, because Möbius transformations map circles to circles.

The round disks for a given $\mathbb{CP}^1$ surface correspond to a family of planes in $\mathbb{H}^3$. Their envelope is a locally convex pleated plane equivariant with respect to a holonomy representation $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$.

Roughly, the pleated plane is the "convex hull boundary" of the $\mathbb{CP}^1$ structure on $\tilde{S}$.

The charts of the $\mathbb{CP}^1$ structure are obtained from the Gauss map of the surface, following normal rays to $\mathbb{CP}^1 = \partial_\infty \mathbb{H}^3$.



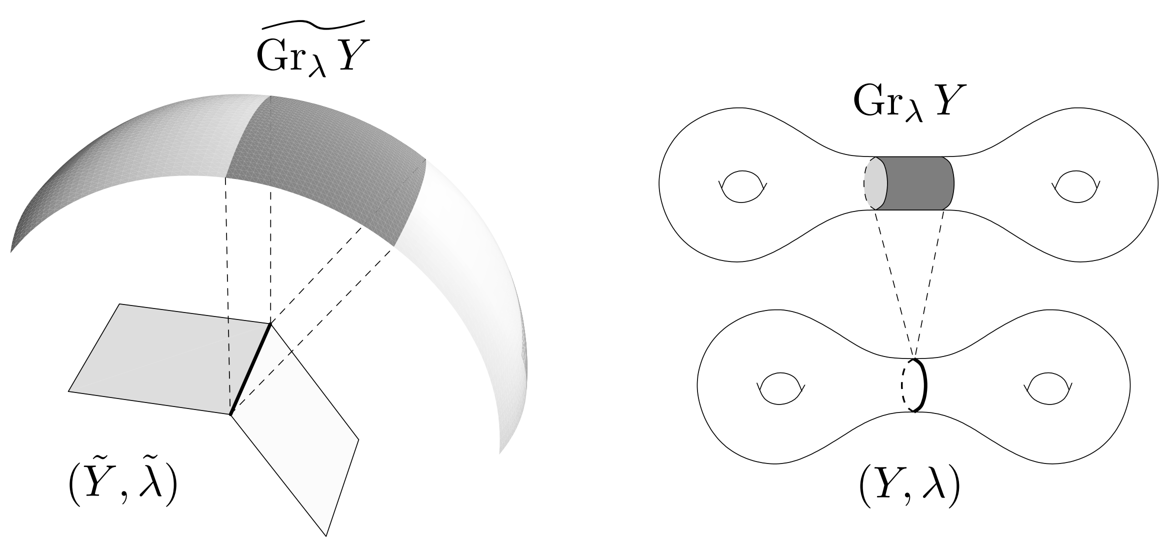
(Actually, the gauss map is defined on the set of unit normal vectors, which is itself a surface.)

Thurston showed that a $\mathbb{CP}^1$ structure is uniquely determined by the associated pleated plane, or equivalently, by its quotient hyperbolic surface Y and its bending lamination λ. (Kamishima-Tan)

Thus $\mathscr{P}(S)$ can be identified with the product of the the PL-manifold of measured laminations and the Teichmüller space of hyperbolic structures.

$$\mathscr{P}(S) \simeq \mathscr{ML}(S) \times \mathscr{T}(S)$$

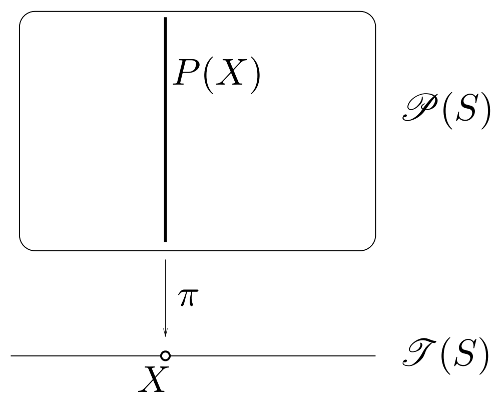
The map Gr : $\mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$ is called **grafting**, because the $\mathbb{CP}^1$ surface is obtained by inserting Euclidean regions along the bending lines of the pleated surface.



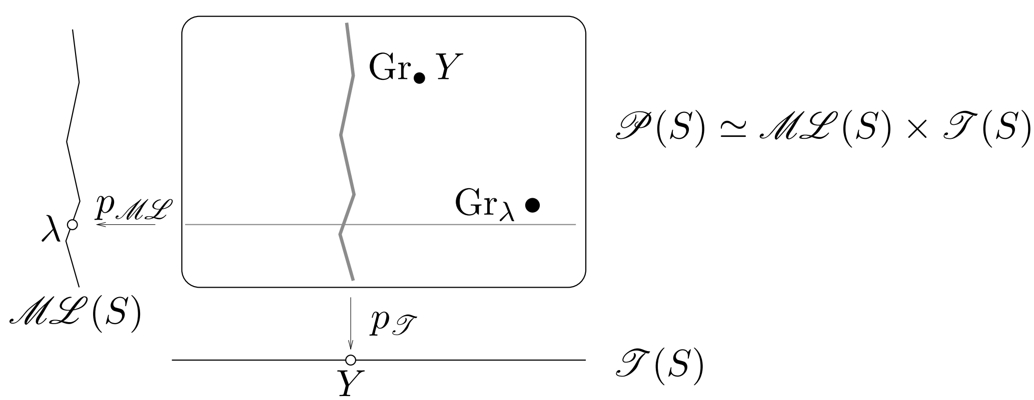
- Comparison -

We now have two models for $\mathscr{P}(S)$:

It is a bundle over Teichmüller space with fibers of constant underlying *complex* structure.



It is the product of the Teichmüller space of *hyperbolic* structures and the space of measured laminations; fibers correspond to fixing some property of the associated pleated plane.



- Questions -

How are these two models related? How do X and ϕ determine Y and λ?

- locally? (infinitesimally?)
- globally? (asymptotically?)

For example, we might ask how a fiber $P(X)$ looks as a subset of $\mathscr{ML}(S) \times \mathscr{T}(S)$ (i.e. the tangent space, projection to a factor, limiting behavior...).

Ultimately these become questions about the grafting maps

$$\mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$$
$$\mathrm{gr} = \pi \circ \mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$$

(That is, $\mathrm{Gr}_\lambda Y$ is a $\mathbb{CP}^1$-surface, with underlying complex structure $\mathrm{gr}_\lambda Y$.)

Specifically,

- What is the derivative of Gr?
- What is the large-scale behavior of Gr?

- Global Results -

For $X \in \mathscr{T}(S)$, define

$$\begin{aligned} M_X &= \{(\lambda, Y) \in \mathscr{ML}(S) \times \mathscr{T}(S) \mid \mathrm{Gr}_\lambda Y \in P(X)\} \\ &= \mathrm{Gr}^{-1}(P(X)) \\ &= \mathrm{gr}^{-1}(X) \end{aligned}$$

So M_X is the set of pairs (λ, Y) representing $\mathbb{CP}^1$ structures with underlying complex structure X in the grafting coordinates for $\mathscr{P}(S)$.

Thm (D; Tanigawa; Scannell-Wolf): The projections $M_X \xrightarrow{p_{\mathscr{ML}}} \mathscr{ML}(S)$ and $M_X \xrightarrow{p_{\mathscr{T}}} \mathscr{T}(S)$ are proper maps of degree 1.

Thus M_X looks like a graph over each factor, at least on a large scale.

The theorem follows from results of Tanigawa and Scannell-Wolf on grafting and the relationship between the two projections:

Thm (D): The closure of M_X in $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$ is topologically a closed ball, and its boundary is the graph of the antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$.

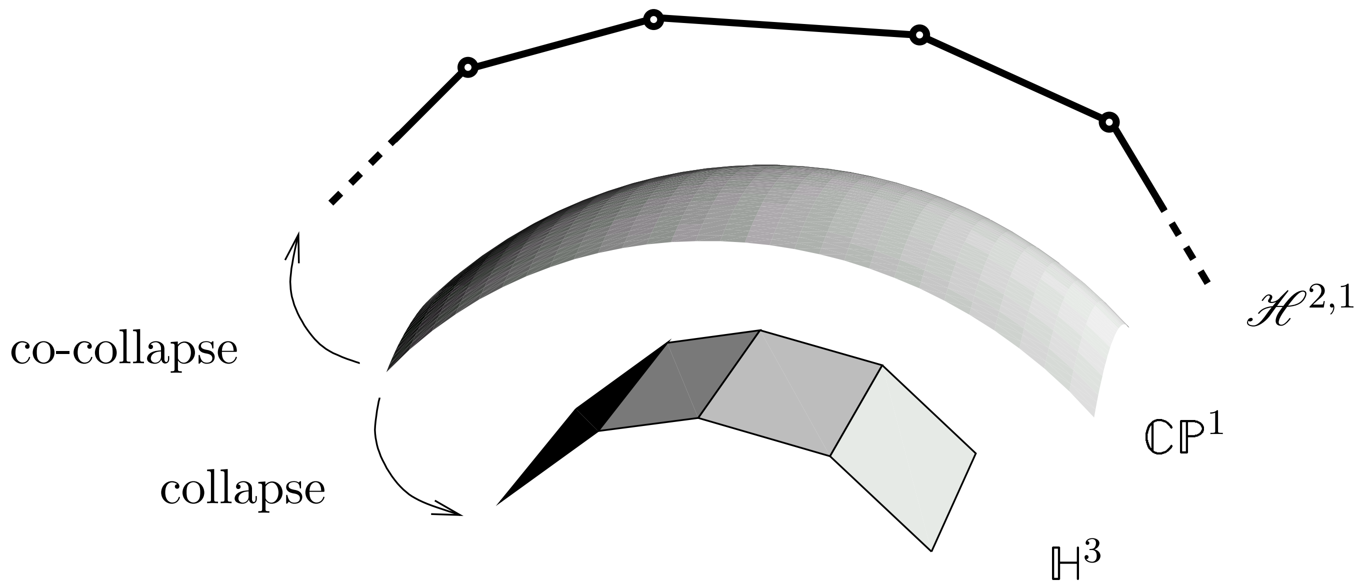
Notes:

1. Here $\overline{\mathscr{ML}(S)}$ is the projective compactification and $\overline{\mathscr{T}(S)}$ is the Thurston compactification; both have boundary $\mathbb{P}\mathscr{ML}(S)$.
2. The antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$ exchanges laminations corresponding to vertical and horizontal trajectories of quadratic differentials on X.

The proof of this result uses properties of two maps associated to a $\mathbb{CP}^1$ structure.

The **collapsing map** is the "nearest-point retraction" to the associated pleated surface in $\mathbb{H}^3$. It collapses the grafted part of $X = \mathrm{gr}_\lambda Y$ onto the associated geodesic lamination in Y.

The **co-collapsing** map sends a point in $\tilde{X} = \widetilde{\mathrm{gr}_\lambda Y}$ to the associated support plane of the pleated surface, which is a point in $\mathscr{H}^{2,1}$, the Lorentz manifold of planes in $\mathbb{H}^3$. The set of such support planes forms the dual tree of λ (typically an $\mathbb{R}$-tree).



The key fact is that both maps are nearly harmonic, i.e. nearly energy-minimizing. (Due to Tanigawa for the collapsing map.)

Combined with the structure theory of harmonic maps between surfaces and from surfaces to trees (Wolf), the closure of M_X can be determined by a geometric limit argument:

- Collapsing and co-collapsing are maps with an exact duality, each is approximately harmonic.
- For a divergent sequence of $\mathbb{CP}^1$ structures on X, energy-normalized limit is a pair of (genuinely) harmonic maps to trees.
- Duality implies antipodal relationship between limit maps.

- Relation to the Schwarzian -

So far we have discussed how X determines the pairs $(\lambda, Y) \in M_X$ (in the large).

How is the Schwarzian derivative related to the grafting coordinates?

Thm (D): Let $\mathrm{Gr}_\lambda Y \in P(X)$ be a $\mathbb{CP}^1$ structure with Schwarzian derivative $\phi \in Q(X)$. Let $\psi \in Q(X)$ be the unique quad. diff. whose horizontal foliation is equiv. to λ. Then

$$\|2\phi + \psi\|_{L^1(X)} = O(\|\psi\|^{\frac{1}{2}}).$$

In particular, the measured foliation of X coming from the Schwarzian (suitably normalized) is asymptotically equal to the one coming from the grafting lamination.

Notes:

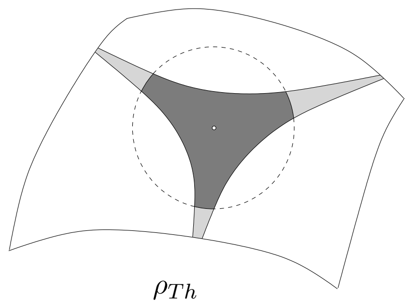
1. The existence of $\psi \in Q(X)$ with any given trajectory structure is a theorem of Hubbard-Masur (Marden-Strebel for multicurves).

2. The implicit constant depends on the moduli of X; since $Q(X)$ is finite dimensional, $L^1(X)$ could be replaced by any norm.

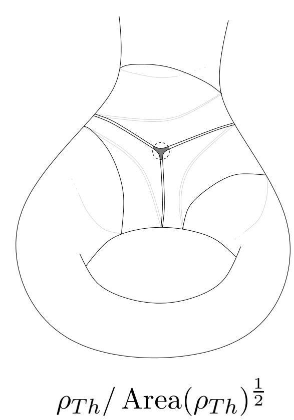
The proof relies on analytic properties of the *Thurston metric*, a conformal metric on $\mathrm{Gr}_\lambda Y$ that combines the hyperbolic metric of Y and the measure of λ. (Kulkarni-Pinkall: metric for higher-dim. Möbius structures)

The Schwarzian derivative is determined by the 2-jet of the Thurston metric. (Osgood-Stowe: interpretation of Schwarzian derivative in terms of conformal metrics; C. Epstein: interpretation as curvature of surface in $\mathbb{H}^3$.)

Bounding the difference between the Schwarzian and the Hubbard-Masur differential amounts to a Sobolev estimate for the Thurston metric, which follows from estimates on its curvature.



ρ_{Th}



$\rho_{Th}/\operatorname{Area}(\rho_{Th})^{\frac{1}{2}}$

For large λ, the Thurston metric on $\mathrm{Gr}_\lambda Y$ is mostly Euclidean, with negative curvature concentrated near a few points. (This gives an L^2 estimate for the Laplacian of the Thurston metric.)

- Holonomy Applications -

The holonomy of a $\mathbb{CP}^1$ structure $Z \in \mathscr{P}(S)$ is a homomorphism

$$\mathrm{hol}(Z) : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$$

well-defined up to conjugation. Thus we have the *holonomy map*

$$\mathrm{hol} : \mathscr{P}(S) \to \mathscr{V}(S) = \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))/\!\!/\mathrm{PSL}_2(\mathbb{C})$$

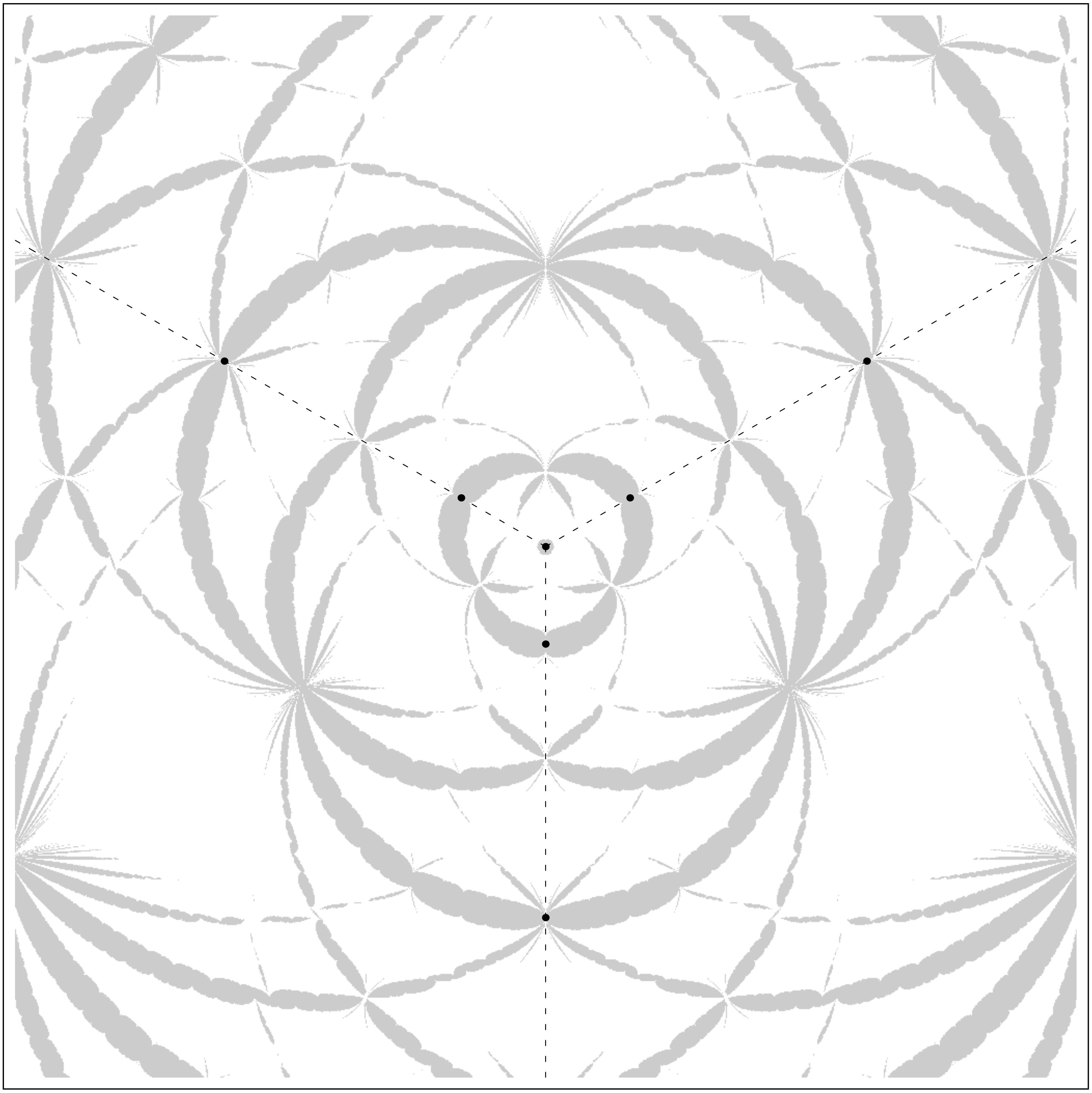
which is a local homeomorphism (Hejhal).

When restricted to a fiber, $\mathrm{hol}_X : P(X) \to \mathscr{V}(S)$ is a proper holomorphic embedding (Gallo-Kapovich-Marden), which intersects the space $\mathscr{QF}$ of quasi-fuchsian representations in countably many "islands" of quasi-fuchsian holonomy (Goldman, Tanigawa).

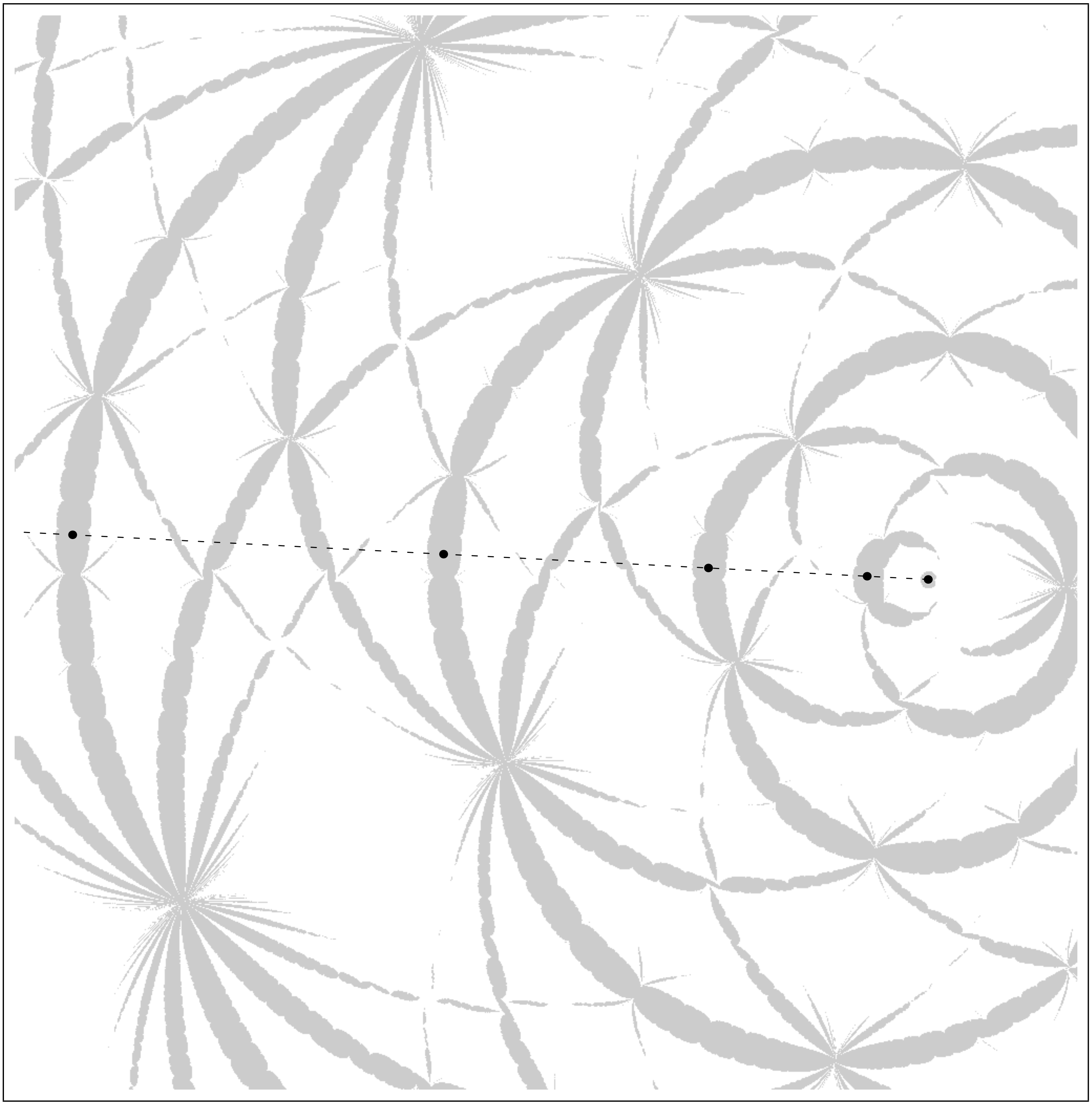
For example, the Bers embedding with basepoint X is one of the connected components of $\mathrm{hol}_X^{-1}(\mathscr{QF})$.

The relation between the Schwarzian and grafting laminations allows other islands to be located, at least approximately, since the holonomy is Fuchsian when λ is 2π-integral. (These are the *Fuchsian centers*.)

Thus the 2π-integral Jenkins-Strebel differentials in $Q(X)$, which form a discrete set with a regular structure, predict the locations of islands of quasi-fuchsian holonomy.



Fuchsian centers for the hexagonal punctured torus.



Fuchsian centers for a punctured torus with no symmetries.

- Infinitesimal Results -

Now we study the small-scale properties of $\mathscr{P}(S)$ and specifically M_X.

Recall $\mathrm{gr} = \pi \circ \mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$ is the conformal grafting map (i.e. graft then forget the $\mathbb{CP}^1$ structure).

Thm (Scannell-Wolf): For each $\lambda \in \mathscr{ML}(S)$, the conformal λ-grafting map $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ is a diffeomorphism.

Scannell-Wolf prove infinitesimal injectivity, which yields the theorem when combined with Tanigawa's result that gr_λ is proper.

Cor: For each $X \in \mathscr{T}(S)$, $M_X \subset \mathscr{ML}(S) \times \mathscr{T}(S)$ is a graph over $\mathscr{ML}(S)$.

Proof: It is the graph of $\lambda \mapsto \mathrm{Gr}_\lambda(\mathrm{gr}_\lambda^{-1}(X))$. $\square$

While $\mathscr{ML}(S)$ has no differentiable structure, the identification between $\mathscr{ML}(S) \times \mathscr{T}(S)$ and $\mathscr{P}(S)$ is "as smooth as possible":

Thm (Bonahon): The grafting map $\mathrm{Gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{P}(S)$ is *tangentiable* (has one-sided derivatives everywhere).

Thus an infinitesimal analysis of the map $\mathrm{gr}_{\bullet} Y : \mathscr{ML}(S) \to \mathscr{T}(S)$ is possible (along the lines of the Scannell-Wolf result).

Thm (D-Wolf): For each $Y \in \mathscr{T}(S)$, the conformal Y-grafting map $\mathrm{gr}_{\bullet} Y : \mathscr{ML}(S) \to \mathscr{T}(S)$ is a (tangentiable) diffeomorphism.

As before this yields a corollary about M_X:

Cor: For each $X \in \mathscr{T}(S)$, $M_X \subset \mathscr{ML}(S) \times \mathscr{T}(S)$ is a graph over $\mathscr{T}(S)$.

As in the Scannell-Wolf grafting theorem, the fact that $\mathrm{gr}_\bullet Y : \mathscr{ML}(S) \to \mathscr{T}(S)$ is a homeomorphism follows from its local injectivity because it is proper.

In fact we show that the infinitesimal injectivity of $\mathrm{gr}_\bullet Y : \mathscr{ML}(S) \to \mathscr{T}(S)$ is a formal consequence of the injectivity of $\mathrm{gr}_\lambda : \mathscr{T}(S) \to \mathscr{T}(S)$ and the Thurston-Bonahon theory of shear-bend coordinates:

Bonahon showed that $\mathrm{gr}_\lambda Y$ has complex-linear derivative with respect to a certain model for (each PL face of) $T_\lambda \mathscr{ML}(S) \oplus T_Y \mathscr{T}(S)$.

In this "shear-bend" model, the complex structure interchanges tangent vectors to $\mathscr{ML}(S)$ and $\mathscr{T}(S)$.

This allows us to turn a failure of local injectivity of $\mathrm{gr}_\bullet Y$ into a failure of local injectivity of gr_λ, which is ruled out by Scannell-Wolf.

(A PDE argument using the Thurston metric, more like that of Scannell-Wolf, may be possible.)

- Summary of Results -

Using grafting, $\mathscr{P}(S)$ is diffeomorphic to the product $\mathscr{ML}(S) \times \mathscr{T}(S)$ (Thurston, Bonahon).

In this model, the fiber $P(X) \subset \mathscr{P}(S)$ with constant underlying complex structure X corresponds to a submanifold $M_X \subset \mathscr{ML}(S) \times \mathscr{T}(S)$ that:

1. is properly embedded,
2. projects diffeomorphically onto each factor,
3. limits to the graph of an involution
$$i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$$
in the boundary of $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$.

Furthermore, the projection of M_X onto $\mathscr{ML}(S)$ is asymptotic to the map that straightens the foliation of X corresponding to the Schwarzian of the $\mathbb{CP}^1$ structure, with an explicit bound on the difference.

In particular:

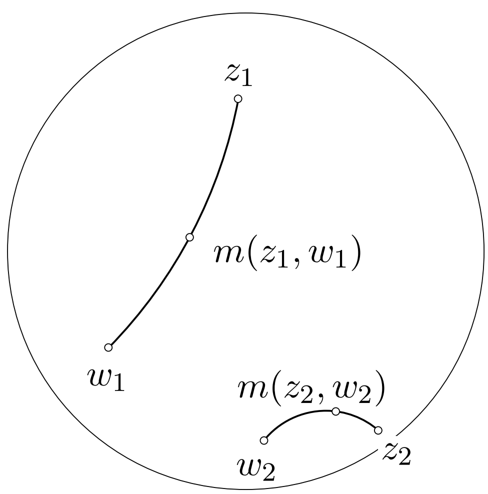
1. While M_X and M_Y are disjoint, their closures in $\overline{\mathscr{ML}(S)} \times \overline{\mathscr{T}(S)}$ always intersect.
2. The set of $X \in \mathscr{T}(S)$ such that M_X has $([\lambda], [\mu]) \in \mathbb{P}\mathscr{ML}(S) \times \mathbb{P}\mathscr{ML}(S)$ in its closure is a Teichmüller geodesic.

- Qualitative Model -

We can construct a rough model for $\mathscr{P}(S)$ and its two coordinate systems based on the results above.

Both $\mathscr{T}(S)$ and $\mathscr{ML}(S)$ are homeomorphic to even-dimensional open balls. We use $\mathbb{B}^{2n} \times \mathbb{B}^{2n}$ as a model for $\mathscr{P}(S) \simeq \mathscr{ML}(S) \times \mathscr{T}(S)$.

View $\mathbb{B}^{2n}$ as the unit ball model of $\mathbb{H}^{2n}$, and let $m : \mathbb{B}^{2n} \times \mathbb{B}^{2n} \to \mathbb{B}^{2n}$ denote the map that associates to (z, w) the midpoint of the hyperbolic geodesic segment joining them.

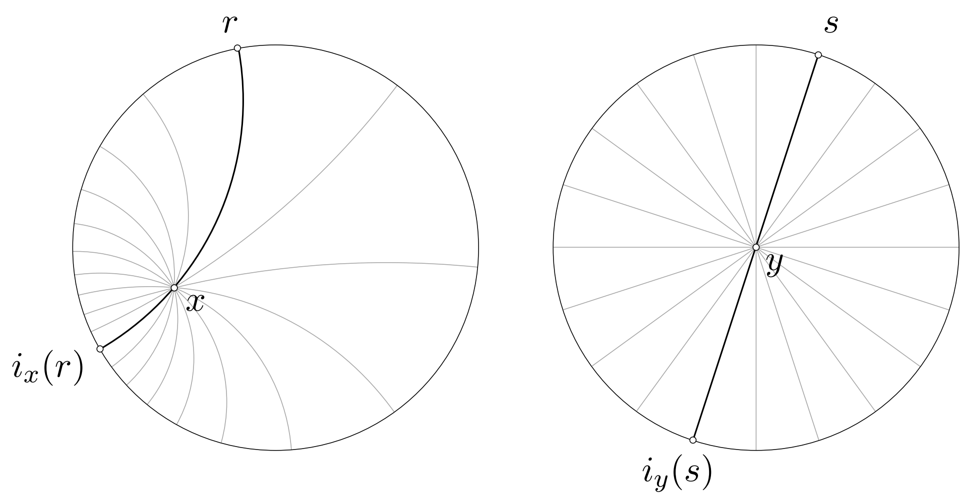


The map m represents the forgetful map $\pi : \mathscr{P}(S) \to \mathscr{T}(S)$ that takes a $\mathbb{CP}^1$ surface to its underlying complex structure.

The fiber of m over x is the set M_x of pairs $(z, w) \in \mathbb{B}^{2n} \times \mathbb{B}^{2n}$ with midpoint x. This represents M_X.

The boundary of M_x in $\overline{\mathbb{B}^{2n}} \times \overline{\mathbb{B}^{2n}}$ is the set of pairs $(r, s) \in \mathbb{S}^{2n-1} \times \mathbb{S}^{2n-1}$ that are endpoints of complete geodesics through x.

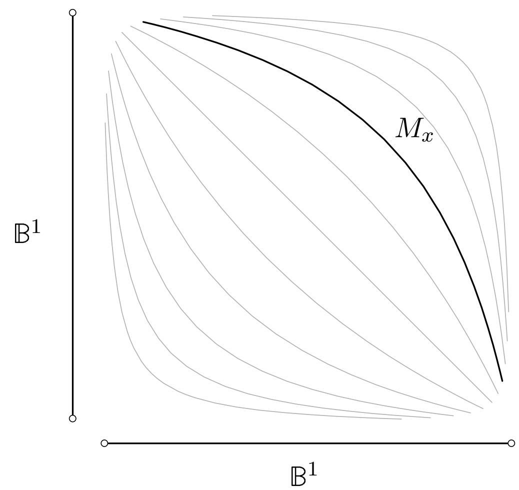
Equivalently, the boundary of M_x is the graph of the *geodesic involution* $i_x : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$ that exchanges endpoints of hyperbolic geodesics through x.



For each $x \in \mathbb{B}^{2n}$, i_x is a fixed-point-free involution of $\mathbb{S}^{2n-1}$; it is analogous to the antipodal involution $i_X : \mathbb{P}\mathscr{ML}(S) \to \mathbb{P}\mathscr{ML}(S)$.

Note: While i_X exchanges foliations corresponding to *Teichmüller* geodesics through X, such geodesics do not necessarily converge in the *Thurston* compactification (cf. Masur, Lenzhen). This is a limitation of the analogy.

The fibers $\{M_x \mid x \in \mathbb{B}^{2n}\}$ of the midpoint map m foliate $\mathbb{B}^{2n} \times \mathbb{B}^{2n}$, much as $\{M_X \mid X \in \mathscr{T}(S)\}$ foliate $\mathscr{ML}(S) \times \mathscr{T}(S)$ with leaves of constant underlying complex structure.



(In this picture, $n = \frac{1}{2}$.)

Note: This analogy compares the conformal grafting map $\mathrm{gr} : \mathscr{ML}(S) \times \mathscr{T}(S) \to \mathscr{T}(S)$ to the midpoint map $m : \mathbb{B}^{2n} \times \mathbb{B}^{2n} \to \mathbb{B}^{2n}$. (?)

The point is not that $\mathrm{gr}_\lambda Y$ is the "midpoint" of λ and Y in any sense, but that if $\mathrm{gr}_\lambda Y$ is fixed, then Y and λ must go to opposite points in $\mathbb{P}\mathscr{ML}(S)$—the common boundary of $\mathscr{T}(S)$ and $\mathscr{ML}(S)$.