Math 442 - Differential Geometry of Curves and Surfaces

Midterm Topic Outline

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This draft of the outline only describes topics that we did not cover in class on March 2. Even for this material, the outline is not guaranteed to be exhaustive. Anything we covered in class or that was in the assigned reading may be on the exam.

(3) Surfaces

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(h) The *inverse function theorem*: If $\phi : U \to V$ is a differentiable map and if $d\phi_p$ is an isomorphism ($\Leftrightarrow$ the matrix of partial derivatives at p is invertible), then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(i) General philosophy: Many ideas from multivariable calculus can be generalized to regular surfaces. Often the generalization is defined like this: Use local coordinates to move everything into $\mathbb{R}^2$, then apply the usual definition for functions of two variables.

(j) *Differentiable functions on surfaces.* A function $f : S \to \mathbb{R}$ on a regular surface can be locally expressed as $f(u, v)$, where (u, v) are local coordinates on S near a point $p = (u_0, v_0)$. If $f(u, v)$ is differentiable (in the multivariable calculus sense) at (u_0, v_0), then we say f is *differentiable at* p.
This does not depend on the coordinate system, since a change of coordinates is differentiable.
If f is differentiable at every point of S, then it is *differentiable.*

(k) *Differentiable maps between surfaces.* A continuous map $\phi : S_1 \to S_2$ between two regular surfaces can be locally expressed as $f(u, v) = (s(u, v), t(u, v))$, where (u, v) are local coordinates on S_1 near $p = (u_0, v_0)$ and (s, t) are local coordinates on S_2 near $\phi(p)$. If $s(u, v)$ and $t(u, v)$ are differentiable at p, then we say ϕ is *differentiable* at p.
If ϕ is differentiable at every point of S_1, then ϕ is *differentiable.*

(l) *Tangent plane.* If $X(u, v)$ is a local parameterization of S, then the span of X_u and X_v at a point p is the *tangent plane of* S *at* p, denoted T_pS.
An alternate definition: Consider the set of all curves in S that pass through p. The set consisting of their tangent vectors at p is T_pS.

(m) *Differential.* A differentiable map $\phi : S_1 \to S_2$ induces a linear map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$, the *differential of* ϕ *at* p. In local coordinates (u, v) near p and (s, t) near $\phi(p)$, we can write $\phi(u, v) = (s(u, v), t(u, v))$. Then the differential has matrix

$$d\phi_p = \begin{pmatrix} \frac{\partial s}{\partial u}(p) & \frac{\partial s}{\partial v}(p) \\ \frac{\partial t}{\partial u}(p) & \frac{\partial t}{\partial v}(p) \end{pmatrix}.$$

(n) The *inverse function theorem for surfaces.* If $\phi : S_1 \to S_2$ is a differentiable map and $d\phi_p$ is an isomorphism, then ϕ is a diffeomorphism *near* p, i.e. there exists a neighborhood U' of p such that $\phi : U' \to V' = \phi(U')$ is a diffeomorphism.

(o) A map $\phi : S_1 \to S_2$ whose differential is an isomorphism at every point need not be injective or surjective.
Examples:
  (i) The inclusion of a small disk by a coordinate chart (not surjective).

   (ii) The plane mapping to the torus by a doubly-periodic parameterization function (not injective).
 (p) Some ways to construct surfaces:
   (i) The cone on the space curve $\alpha(t)$ is parameterized by $X(s,t) = t\alpha(s)$.
   (ii) The surface of rotation of a plane curve $(\varphi(t), \psi(t))$ is parameterized by $X(t, \theta) = (\varphi(s)\cos(\theta), \varphi(s)\sin(\theta), \psi(s))$.
   (iii) The surface of rotation of a circle that does not intersect the y axis is a *circular torus.*
   (iv) A surface that contains a line segment through each of its point is *ruled.* Such a surface can be parameterized by $X(s,t) = \alpha(s) + t\beta(s)$.

(4) First and second fundamental forms
 (a) Restricting the inner product of $\mathbb{R}^3$ makes T_pS into an *inner product space.* The associated quadratic form is the *first fundamental form*, denoted I_p. Thus $I_p(w)$ is the squared length of w (as a vector in $\mathbb{R}^3$).
 (b) In the basis X_u, X_v for T_pS given by a local parameterization, the matrix of I_p is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ where $E = \langle X_u, X_u\rangle$ $F = \langle X_u, X_v\rangle$ $G = \langle X_v, X_v\rangle$.
 In other words, we have "$I = \langle dX, dX\rangle$".
 (c) The length of a curve $\alpha(t) = (u(t), v(t))$ on S is given by

$$\int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

 Note that Eu'^2 means $E(u(t), v(t))\,(u'(t))^2$, and similarly for the other terms.
 (d) The area of a region Ω contained in a local coordinate chart (u, v) is given by

$$\iint_\Omega \sqrt{EG - F^2}dudv.$$

 Note that when S is contained in $\mathbb{R}^2$, this is the usual formula for change of variables, and $\sqrt{EG - F^2}$ is the Jacobian of the transformation.
 (e) The angle θ between vectors $w_1 = aX_u + bX_v$ and $w_2 = cX_u + dX_v$ satisfies

$$\cos(\theta) = \frac{\langle w_1, w_2\rangle}{|w_1|\,|w_2|} = \frac{Eac + F(ad + bc) + Gbd}{\sqrt{(Ea^2 + 2Fab + Gb^2)(Ec^2 + 2Fcd + Gd^2)}}$$

 (f) A map between surfaces whose differential preserves length of vectors is a (local) *isometry.* If the differential preserves angles, then the map is *conformal.* Note than an isometry is conformal.
 (g) An *orientation* of a surface is a choice of a unit normal vector at each point in such a way that the resulting map $N : S \to S^2$ is continuous. Here $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. If a surface S has an orientation, then it has exactly two, and we say S is *orientable.*
 (h) When parameterizing an oriented surface, we always choose $X(u, v)$ so that $X_u \wedge X_v$ is a positive multiple of the unit normal, i.e.

$$N(u, v) = \frac{X_u \wedge X_v}{|X_u \wedge X_v|}$$

 (i) The map N is called the *Gauss map* of the surface. The differential of the Gauss map is self-adjoint with respect to I_p.
 (j) The *second fundamental form* is the quadratic form II_p on T_pS defined by $II_p(w) = -\langle dN_p(w), w\rangle = \langle \frac{\partial^2 X}{\partial w^2}, N(p)\rangle$. So II_p is the normal component of the acceleration of a path in S with tangent vector w. One could summarize this definition as "$II = -\langle dX, dN\rangle = \langle d^2X, N\rangle$".
 (k) The eigenvalues of $-dN_p$ are the *principal curvatures* of S at p, denoted k_1, k_2. The associated eigenspaces are the *principal directions.*

(l) The product of the principal curvatures is the *Gaussian curvature* $K(p) = k_1(p)k_2(p) = \det(dN_p)$.

(m) The average of the principal curvatures is the *mean curvature* $H(p) = \frac{1}{2}(k_1(p) + k_2(p)) = \mathrm{tr}(dN_p)$.

(n) In local coordinates, the matrix of II_p is given by $\begin{pmatrix} e & f \\ f & g \end{pmatrix}$ where:

$$\begin{aligned} e &= \langle X_{uu}, N\rangle = -\langle X_u, N_u\rangle \\ f &= \langle X_{uv}, N\rangle = -\langle X_u, N_v\rangle = -\langle X_v, N_u\rangle \\ g &= \langle X_{vv}, N\rangle = -\langle X_v, N_v\rangle \end{aligned}$$

(o) This is different from the matrix of dN_p, unless X_u and X_v are orthonormal. In general, we have $dN_p = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$.

(p) Using the formula for N in terms of X_u and X_v gives the convenient formula

$$e = \frac{1}{\sqrt{EG - F^2}} \det(X_u \quad X_v \quad X_{uu})$$

and similarly for f and g, replacing only the second derivative term with X_{uv} or X_{vv}, respectively.

(q) Using the formula for dN_p, we have

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{Eg - 2Ff + Gg}{2(EG - F^2)}$$

and the principal curvatures are the roots of the polynomial $\lambda^2 - 2H\lambda + K$.

(r) If $\alpha(s)$ is a curve contained in S, then the length of the projection of $\alpha''(s)$ onto $N(\alpha(s))$ is the *normal curvature* of α, denoted k_N. The normal curvature at $\alpha(s)$ only depends on $\alpha'(s)$, and is given by $II_{\alpha(s)}(\alpha'(s))$. Here we assume $\alpha(s)$ is parameterized by arc length.

(s) The principal curvatures at p are the extreme values of the normal curvature as α' varies over all unit tangent vectors at p.

(t) Classification of points on a surface:
- If $K(p) > 0$, then p is an *elliptic point.*
- If $K(p) = 0$ but dN_p is nonzero, then p is a *parabolic point.*
- If $K(p) = 0$ and dN_p is zero, then p is a *planar point.*
- If $K(p) < 0$, then p is a *hyperbolic point.*
- If $k_1(p) = k_2(p)$ (or equivalently, $H(p)^2 = K(p)$), then p is an *umbilic point.*

(u) Typical examples:
- Every point on the unit sphere is elliptic and umbilic.
- Every point on a cylinder is parabolic.
- Every point on a plane is planar
- The point $(0,0,0)$ on $\{z = (x^2 + y^2)^2\}$ is planar.
- The point $(0,0,0)$ on the "saddle" $\{z^2 = x^2 - y^2\}$ is hyperbolic.
- If $f''(x) > 0$, then every point on the surface of rotation of f is hyperbolic.
- The point $(0,0,0)$ on the circular paraboloid $\{z = x^2 + y^2\}$ is umbilic.

(v) A curve in S whose tangent vector at each point is a principal direction is a *line of curvature.*

(w) Special cases:
- If $F = 0$, then the horizontal and vertical lines in the uv plane correspond to orthogonal curves in S.
- If $F = f = 0$, then the principal curvatures are e/E and g/G, the principal directions are X_u and X_v, and the horizontal and vertical lines in the uv plane correspond to lines of curvature in S.