Math 442 / Differential Geometry of Curves and Surfaces / David Dumas

Problem from lecture 36 (November 15, 2010)

(Lec36 P1) Consider the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the isothermal metric $\lambda(x, y) = \frac{1}{y}$. (This means the first fundamental form has coefficients $E = G = \frac{1}{y^2}$ and $F = 0$.)

(a) Write out the geodesic equations for this metric in terms of the coordinates $\gamma(t) = (x(t), y(t))$ of a curve in $\mathbb{H}$.

(b) Show that the upper half of any vertical line $x = x_0$ in $\mathbb{R}^2$ is a geodesic, with the parameterization $\gamma(t) = (x_0, e^t)$. Conclude that these are all of the geodesics in $\mathbb{H}$ that have a vertical tangent at some point.

(c) Given a point (x, y) in $\mathbb{H}$ and a tangent vector (x', y') with $x' \neq 0$, there is a unique circle in $\mathbb{R}^2$ centered on the x-axis that contains (x, y) and which has (x', y') as a tangent vector. Show that the square of the radius of this circle is

$$R(x, y, x', y') = y^2 \left(1 + \left(\frac{y'}{x'}\right)^2\right).$$

(d) Given a curve $\gamma(t) = (x(t), y(t))$ in $\mathbb{H}$ we can consider $R(t) = R(x(t), y(t), x'(t), y'(t))$ as a function of t. Show that this function is constant if γ is a geodesic whose tangent vector at some point is not vertical. Conclude that geodesics of $\mathbb{H}$ that are not vertical lines are contained in circles centered on the x-axis.

(e) Show that $\gamma(t) = (\tanh(t), \operatorname{sech}(t))$ is a parameterization of the upper half of the unit circle that is geodesic in $\mathbb{H}$.

(Recall $\tanh(t) = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$ and $\operatorname{sech}(t) = \dfrac{2}{e^t + e^{-t}}$.)