Math 442 / Differential Geometry of Curves and Surfaces / David Dumas

Problems from lecture 9 (September 13, 2010)

A set $\Omega \subset \mathbb{R}^n$ is *convex* if for any $p, q \in \Omega$, the line segment with endpoints p, q is contained in Ω. In this problem set Ω will denote an open convex subset of $\mathbb{R}^2$.

In solving the problems below, you may appeal to the following basic properties of the boundary curve C of an open convex set Ω without proof:

- The boundary C is a simple closed curve to which Crofton's formula applies, i.e. the intersection counting function n_C is integrable and its integral is $2L$, where L is the length of C.
- A line in the plane intersects C in either zero or two points, except for a set of lines with area zero. (If C were a smooth curve, then this set of exceptions would consist of the tangent lines to C.)

Be careful to indicate exactly where and how you use these properties.

(Lec9 P1) The *horizontal width* of a convex set $\Omega \subset \mathbb{R}^2$ is the difference between the supremum and infimum of the x-coordinates of points in Ω. The *θ-width* of Ω, denoted by $w_\theta(\Omega)$, is the horizontal width of the set obtained by rotating Ω clockwise by angle θ.

Show that if C is the boundary curve of Ω then

$$L(C) = \int_0^\pi w_\theta(\Omega) d\theta.$$

(Lec9 P2) Let Ω_1, Ω_2 be open convex sets in $\mathbb{R}^2$ with $\Omega_1 \subset \Omega_2$. Let C_i denote the boundary curve of Ω_i. Show that $L(C_1) \leq L(C_2)$.

(Lec9 P3) If Ω is an open convex set in $\mathbb{R}^2$, let $N_\varepsilon(\Omega)$ denote the set of all points in $\mathbb{R}^2$ whose distance to Ω is less than ε, i.e.

$$N_\varepsilon(\Omega) = \{p \in \mathbb{R}^2 \mid \text{There exists } q \in \Omega \text{ such that } |p - q| < \varepsilon\}.$$

Let C,C_ε denote the boundary curves of Ω and $N_\varepsilon(\Omega)$, respectively. Show that $L(C_\varepsilon) = L(C) + 2\pi\varepsilon$.