Math 442 / David Dumas / Fall 2010

Midterm Solutions

(1) Prove that if two regular surfaces intersect at only one point, then they have the same tangent plane at that point. (That is, if \( S_1 \cap S_2 = \{p\} \) then \( T_pS_1 = T_pS_2 \).

Solution. It is enough to show that if two regular surfaces \( S_1, S_2 \) intersect at \( p \) and \( T_pS_1 \neq T_pS_2 \), then the set \( S_1 \cap S_2 \) contains more than one point. In fact we will show that \( S_1 \cap S_2 \) contains a curve through \( p \).

For \( i = 1, 2 \), represent \( S_i \) in a neighborhood of \( p \) as \( \{(x, y, z) \mid f_i(x, y, z) = 0\} \) where \( f_i \) is a differentiable function with 0 as a regular value (so in particular \( \nabla f_i(p) \neq 0 \)).

The tangent plane of \( S_i \) at \( p \) is the plane through \( p \) with normal vector \( \nabla f_i(p) \). Since the tangent planes are different, the vectors \( \nabla f_1(p) \) and \( \nabla f_2(p) \) are linearly independent.

Let \( f_3(x, y, z) \) be a differentiable function defined in a neighborhood of \( p \) such that \( f_3(p) = 0 \) and \( \{\nabla f_1(p), \nabla f_2(p), \nabla f_3(p)\} \) is a basis of \( \mathbb{R}^3 \). (For example, complete the linearly independent set \( \{\nabla f_1(p), \nabla f_2(p)\} \) to a basis by adding a vector \( v \), and then let \( f_3(q) = (p - q) \cdot v \).

Define \( F(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \). Then \( F(p) = 0 \) and the rows of \( dF_p \) are linearly independent, so by the inverse function theorem \( F \) is a diffeomorphism from a neighborhood of \( p \) to a neighborhood of \((0, 0, 0)\). Let \( G \) denote the inverse of this map. Then for all \( t \in \mathbb{R} \) with \( |t| \) sufficiently small, \( G(0, 0, t) \) is defined and lies on both \( S_1 \) and \( S_2 \) (since \( f_i(G(0, 0, t)) = 0 \) for \( i = 1, 2 \)). Since \( G \) is a diffeomorphism, all of the points obtained this way are distinct. This shows that \( S_1 \cap S_2 \) is infinite.

Comment. The intuition behind this solution is the following: Two distinct planes through a point \( p \) in \( \mathbb{R}^3 \) intersect in a line. Locally, regular surfaces are well-approximated like their tangent planes, so if \( T_pS_1 \neq T_pS_2 \), then \( S_1 \cap S_2 \) is approximately a line. In fact, \( S_1 \cap S_2 \) is a regular curve near \( p \) whose tangent line at \( p \) is \( T_pS_1 \cap T_pS_2 \).

(2) Determine the set of all positive real numbers \( A \) such that the equation
\[
(x + y + z)^3 = A \left( x^3 + y^3 + z^3 \right)
\]
defines a regular surface in \( \mathbb{R}^3 - \{(0, 0, 0)\} \).

Solution. Let \( F(x, y, z) = (x + y + z)^3 - A(x^3 + y^3 + z^3) \). We want to know when \( F^{-1}(0) \) is a regular surface in \( \mathbb{R}^3 - \{(0, 0, 0)\} \). Note that \( F \) is symmetric in \( x, y, \) and \( z \). We first determine the critical points of \( F \). We have
\[
\frac{\partial F}{\partial x} = 3 \left[ (x + y + z)^2 - Ax^2 \right]
\]
\[
\frac{\partial F}{\partial y} = 3 \left[ (x + y + z)^2 - Ay^2 \right]
\]
\[
\frac{\partial F}{\partial z} = 3 \left[ (x + y + z)^2 - Az^2 \right]
\]
so critical points are defined by \( x^2 = y^2 = z^2 = \frac{1}{3}(x + y + z)^2 \). In particular, \( x, y, \) and \( z \) are equal up to sign at any critical point.
Consider the case \( x = y = z = s \). In order for this to be a critical point we must have \( A s^2 = (3s)^2 \), so when \( A \neq 9 \), the only critical point on this line is \((0,0,0)\). If however \( A = 9 \), then every point on the line \( x = y = z \) is critical.

Now consider \( x = y = -z = s \). In order for this to be a critical point we must have \( A s^2 = (s + s - s)^2 = s^2 \), so when \( A \neq 1 \) the only critical point on this line is \((0,0,0)\). If however \( A = 1 \), then every point on the line \( x = y = -z \) is critical.

By symmetry we get a similar conclusion for the cases \( x = -y = z \) and \( -x = y = z \), and to summarize:

- If \( A \notin \{1, 9\} \), then \( F \) has no critical points other than \((0,0,0)\).
- If \( A = 9 \), then the critical set of \( F \) is the line \( x = y = z \).
- If \( A = 1 \), then the critical set of \( F \) is the union of the three lines \( x = y = -z \), \( x = -y = z \), \( -x = y = z \).

We immediately conclude that for \( A \notin \{1, 9\} \), zero is a regular value of \( F(x, y, z) \) on \( \mathbb{R}^3 - \{(0,0,0)\} \) and \( F = 0 \) defines a regular surface. It remains to analyze the cases \( A = 1 \) and \( A = 9 \) separately.

**Case \( A = 1 \):** We have
\[
F(x, y, z) = (x + y + z)^3 - x^3 - y^3 - z^3
= 3x^2y + 3xy^2 + 3x^2z + 3y^2z + 3xyz + 3yz^2 + 6xyz
= 3(x + y)(x + z)(y + z)
\]
So \( F^{-1}(0) \) is the union of three distinct planes that meet at \((0,0,0)\). This is not a regular surface, because near a line of intersection of two of these planes (say, in an arbitrarily small neighborhood of \((1,−1,0)\)) the set does not project injectively onto any of the coordinate planes.

**Case \( A = 9 \):** Suppose \( S = F^{-1}(0) \) were a regular surface in \( \mathbb{R}^3 - \{(0,0,0)\} \). In this case \( F(s,s,s) = (3s)^3 - 9(3s^3) = 0 \) so the entire line \( \ell = \{(x,y,z) \mid x = y = z\} \) is contained in \( S \). Therefore at any point \( p \in \ell \), the tangent plane \( T_p S \) must contain \( \ell \). The cyclic permutation \( (x,y,z) \mapsto (y,z,x) \) rotates \( \mathbb{R}^3 \) around \( \ell \) by angle \( 2\pi/3 \), but this permutation does not affect the value of \( F \) so it preserves \( S \) and fixes every point of \( \ell \). Therefore, the tangent plane to \( S \) at \( p \in \ell \) must be invariant under this rotation.

Since no plane in \( \mathbb{R}^3 \) containing \( \ell \) is invariant under rotation by \( 2\pi/3 \) around \( \ell \), this is a contradiction, and \( S \) is not a regular surface.

**Summary.** The equation \( (x + y + z)^3 = A(x^3 + y^3 + z^3) \) defines a regular surface in \( \mathbb{R}^3 - \{(0,0,0)\} \) for all real numbers \( A \) except \( A = 1 \) and \( A = 9 \).

(3) (a) Define the torsion function \( \tau \) of a space curve.

**Solution.** Parameterize the curve by arc length and let \( t(s) = \alpha'(s) \), \( n(s) = t'(s)/|t'(s)| \), and \( b(s) = t(s) \wedge n(s) \). Then the torsion \( \tau(s) \) is the real-valued function such that \( b'(s) = \tau(s)n(s) \) for all \( s \).

(b) Let \( \alpha : I \to \mathbb{R}^3 \) denote a regular parameterized space curve without inflection points. Show that \( \alpha(I) \) lies in a plane if and only if the torsion of \( \alpha \) is identically zero.

**Solution.** If the curve lies in a plane \( P \), then all of its derivatives are parallel to that plane. Therefore \( t(s) \) and \( n(s) \) are parallel to \( P \), hence they span it, and \( b(s) \) is a unit normal vector to \( P \). There are two such unit normals, but by continuity,
$b(s)$ can only assume one of these values. So $b(s)$ is a constant function, and $b'(s) = 0$. This gives $\tau(s) = 0$.

Conversely, suppose the torsion is identically zero. Then $b(s)$ is a constant function; let $N$ denote its value. Then for all $s$, we have $t(s) \cdot N = n(s) \cdot N = 0$. Consider the real-valued function $f(s) = (\alpha(s) - \alpha(s_0)) \cdot N$. Then $f(s_0) = 0$ and using $t(s) \cdot N = 0$ gives $f'(s) = 0$. Therefore the function $f(s)$ is identically zero, which shows that $\alpha$ is contained in the plane \{ $p \mid (p - \alpha(s_0)) \cdot N = 0$ \}.

(4) (a) Define the curvature function $\kappa$ of a plane curve.

**Solution.** Parameterize the curve by arc length and let $t(s) = \alpha'(s)$. Define $n(s)$ to be the unit vector such that $t(s) \cdot n(s) = 0$ and so that the ordered basis $(t(s), n(s))$ is positively oriented. Then the curvature $\kappa(s)$ is the real-valued function such that $t'(s) = \kappa(s)n(s)$.

(b) Determine the curvature function of the cycloid

$$\alpha(t) = (at - b\sin(t), a - b\cos(t))$$

where $a, b \in \mathbb{R}$ are constants and $a \neq 0$.

**Solution.** Note that the given curve is not parameterized by arc length. Up to sign the curvature is given by

$$\left\| \frac{\alpha'(t) \wedge \alpha''(t)}{|\alpha'(t)|^3} \right\|.$$

The correct sign (taking into account the definition of curvature for a plane curve) is given by replacing the numerator with $\det \left( \begin{array}{c} \alpha'(t) \\ \alpha''(t) \end{array} \right)$. We calculate:

$$\alpha'(t) = (a - b\cos(t), b\sin(t))$$
$$\alpha''(t) = (b\sin(t), b\cos(t))$$
$$|\alpha'(t)|^2 = a^2 + b^2 - 2ab\cos(t)$$
$$\det \left( \begin{array}{c} \alpha'(t) \\ \alpha''(t) \end{array} \right) = b(a\cos(t) - b)$$

and therefore

$$\kappa(t) = \frac{b(a\cos(t) - b)}{(a^2 + b^2 - 2ab\cos(t))^{3/2}}.$$

Note that if $a = b$, the curvature is not defined for $t \in 2\pi\mathbb{Z}$. 
The surface \((x + y + z)^3 = A(x^3 + y^3 + z^3)\) for several values of \(A\).

Several cycloids.