Math 569 / David Dumas / Spring 2017

Exercises

This list was last updated on 2017-03-27.

Note: The problems on the list may be occasionally updated with clarification or corrections, but the numbering will never change. Make sure to refer to the problems by number when submitting a solution.

(1) A non-orientable closed surface can always be represented as a sphere with k crosscaps attached, for some $k > 0$; let us denote this surface by N_k. (Here *attaching a crosscap* means removing a disk and gluing a Möbius strip in its place.) In particular, gluing a single crosscap into the orientable surface S_g results in a surface homeomorphic to N_k for some k. Determine k as a function of g, and give a detailed pictorial description of the process in which "handles become crosscaps".

(2) Provide details on the following aspects of the classification of surfaces that are omitted from the proof of Theorem 2.2.13 in the text.
   (a) The edge deletion and edge collapse operations in the first steps of the proof preserve the filling property of the ribbon graph.
   (b) The polygon surgeries described in the last step of the proof correspond to changes in the ribbon graph. (Make explicit what these steps look like from the ribbon graph perspective.)

(3) Show that gluing opposite sides of a polygon with $4g$ or $4g+2$ sides gives a surface of genus g. Do these gluing patterns correspond to ribbon graphs? If so, identify the ribbon graph and the steps (as in Thm 2.2.13) that transform it to the standard form Γ_g.

(4) (This exercise is about correcting some details from section 2.4.2 in the text.) Consider the free monoid on the set I indexing an open cover $\mathscr{U}$ of a space. Define a *loop* based at $i_0 \in I$ to be an element of this monoid satisfying a certain condition, adapting Definition 2.4.4 in the text. If the "inverse" of a loop is defined by reversing the order of symbols in the word, determine an equivalence relation on the submonoid of loops based at i_0 so that the quotient is a group, and is isomorphic to $\pi_1(N_1(\mathscr{U}))$ where $N_1(\mathscr{U})$ is the 1-skeleton of the nerve of $\mathscr{U}$. Then, describe how to enlarge this equivalence relation using the 2-cells of the nerve so as to give a quotient isomorphic to $\pi_1(N(\mathscr{U}))$.

(5) Let Γ be a ribbon graph with edge set E and vertex set V. A *ribbon graph automorphism* is a pair of bijections $f_E : E \to E$ and $f_V : V \to V$ that are compatible with the source and destination maps and which preserve the cyclic orientation on the stars coming from the ribbon structure. State this definition precisely, and the compute the ribbon graph automorphism group of Γ_g and of the ribbon graph corresponding to identifying opposite sides of a $4g+2$-gon. ("Compute" in this problem means give generators and relations for the automorphism group.)

(6) Let f be an automorphism of a ribbon graph Γ. Let $S = S_\Gamma$.
   (a) Show that action of f on the edge set induces a group automorphism $f_* : \pi_1^\Gamma(S) \to \pi_1^\Gamma(S)$ of the ribbon fundamental group.

(b) Construct an orientation-preserving homeomorphism $F : S \to S$ such whose induced map on the fundamental group is f_*.
(c) Construct an example in each genus $g > 0$ of a nontrivial ribbon graph automorphism f such that f_* is the identity map of $\pi_1 S$.

(7) Prove proposition 2.5.25 in the text (i.e. that the dual pairing for ribbon cochains induces a well-defined, nondegenerate pairing on cohomology).

(8) (From the text.) Show that every vector bundle on the unit interval $[0,1]$ is trivial.

(9) (a) Give an example of a nontrivial vector bundle $\mathscr{L}$ such that $\mathscr{L} \oplus \mathscr{L}$ is trivial.
(b) Give an example of a nontrivial vector bundle $\mathscr{L}$ such that $\mathscr{L} \otimes \mathscr{L}$ is trivial.
(c) Let's say that a continuous map $f : N \to M$ is *very surjective* if every map homotopic to f is surjective. Give an example of a nontrivial vector bundle $\mathscr{L}$ over some base M and very surjective map $f : N \to M$ such that $f^*(\mathscr{L})$ is trivial.

(10) Consider the surface S obtained by identifying opposite sides of a polygon with $2n$ sides. Thinking of the polygon as a regular one, rotation of the polygon by angle π/n induces a homeomorphism $T : S \to S$ which has $T^{2n} = \mathrm{id}_S$. For which values of n can S be embedded in $\mathbb{R}^3$ in such a way that T is the restriction to this embedding of a rigid motion? (A rigid motion means any map $\mathbb{R}^3 \to \mathbb{R}^3$ that can be written in the form $f(x) = Ax + b$ where $b \in \mathbb{R}^3$ and where A is an orthogonal matrix.)

(11) Verify that the dual connection ∇^* described in section 3.3.2 satisfies the Leibniz rule.

(12) Prove that there is at most one connection $\phi^*\nabla$ on the pullback bundle $\phi^*\mathscr{L}$ which satisfies equation (3.1) of section 3.3.2; here $\pi : \mathscr{L} \to M$ is a vector bundle, $\phi : N \to M$ a smooth map, and ∇ a connection on $\mathscr{L}$. That is, prove the uniqueness part of the construction of the pullback connection. (Note that the proof of *existence* sketched in this part of the text uses uniqueness.)

(13) Let $\mathscr{L}$ be a line bundle over M. Show that $\mathrm{Hom}(\mathscr{L},\mathscr{L})$ is a trivial line bundle. Furthermore, show that $\mathrm{Hom}(\mathscr{L},\mathscr{L})$ is *canonically trivial* in the following sense: A local trivialization ϕ of $\mathscr{L}$ over $U \subset M$ induces a local trivialization $\tilde{\phi}$ of $\mathrm{Hom}(\mathscr{L},\mathscr{L})$. Show that there is a global trivialization $\Phi : \mathrm{Hom}(\mathscr{L},\mathscr{L}) \to M \times \mathbb{R}$ such that for *any* local trivialization ϕ of $\mathscr{L}$ over U we have $\tilde{\phi} = \Phi|_{\pi^{-1}(U)}$.

(14) Describe the Möbius strip μ as a line bundle over $S^1 = \mathbb{R}/\mathbb{Z}$ using a covering of S^1 by trivializing intervals U, V where $U \cap V$ has two connected components. (Identify U,V with intervals in $\mathbb{R}$ that map diffeomorphically to them under the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.) Then determine the compatibility conditions required of a pair of 1-forms $A_U \in \Omega^1(U)$ and $A_V \in \Omega^1(V)$ in order that the local connections $D + A_U$ on $U \times \mathbb{R}$ and $D + A_V$ on $V \times \mathbb{R}$ define a connection on μ.

(15) Generalizing the previous exercise, give an atlas of local trivializations of the tangent bundle of S^2 using trivializing open sets U, V that are, respectively, open neighborhoods of the upper and lower hemispheres. In this atlas, use stereographic projections $(S^2 \setminus \{0,0,-1\}) \to \mathbb{R}^2$ and $(S^2 \setminus \{0,0,1\}) \to \mathbb{R}^2$ to identify U and V with open sets in the plane. Then determine the compatibility condition required of a pair of 1-forms $A_U \in \Omega^1(U, \mathrm{End}(\mathbb{R}^2))$ and $A_V \in \Omega^1(V, \mathrm{End}(\mathbb{R}^2))$ in order to define a connection on TS^2.

(16) Verify that the local expression of the curvature tensor R^∇ is $DA + \frac{1}{2}[A \wedge A]$, where $D+A$ is corresponding local expression of ∇.

(17) Consider the trivial $\mathbb{R}^2$ bundle over S^1 with the connection $\nabla = D + A$ where $A = \left(\begin{smallmatrix} 0 & \alpha \\ -\alpha & 0 \end{smallmatrix}\right)$ for $\alpha \in \Omega^1(S^1, \mathbb{R})$.
 (a) Show that this is a $SO(2)$-connection.
 (b) When is this connection flat?
 (c) When is this connection gauge-equivalent to the trivial connection?
 (d) Calculate the holonomy representation of this connection.

(18) Show that if a connection ∇ is locally trivial, and if γ_0 and γ_1 are homotopic paths rel endpoints, then $\mathrm{Hol}^\nabla(\gamma_0) = \mathrm{Hol}^\nabla(\gamma_1)$. (That is, fill in the details of the sketch given in lecture.)

(19) Let $\gamma_s(t)$ be a smooth family of paths in M from x_0 to x_1. That is, suppose $(s,t) \to \gamma_s(t)$ is a smooth map $[0,1] \times [0,1] \to M$ with $\gamma_s(0) = x_0$ and $\gamma_s(1) = x_1$ for all s. Let $\mathscr{L}$ be a vector bundle over M and ∇ a connection on $\mathscr{L}$. Define

$$H_s = \mathrm{Hol}^\nabla(\gamma_s) \in \mathrm{Hom}(\mathscr{L}_{x_0}, \mathscr{L}_{x_1}).$$

Show that

$$\frac{d}{ds} H_s = \left(\int_0^1 R^\nabla \left(\frac{\partial \gamma_s(t)}{\partial t}, \frac{\partial \gamma_s(t)}{\partial s} \right) dt \right) H_s$$

The rough interpretation of this statement is: *Under a small change in a path x_0 to x_1, say from γ_0 to γ_ε, the change in holonomy is given by integration of the 2-form R^∇ over the "strip" $\{\gamma_s(t) \,|\, 0 \le s \le \varepsilon, 0 \le t \le 1\}$.*

(20) Use the previous exercise to prove that the holonomy of a flat connection depends only on the homotopy class of a path rel endpoints.

(21) Let $\pi : \mathscr{L} \to M$ be a vector bundle of rank n. A submanifold $\mathscr{F} \subset \mathscr{L}$ is a *subbundle of $\mathscr{L}$ of rank r* if each intersection $\mathscr{L}_x \cap \mathscr{F}$ is a subspace of $\mathscr{L}_x$ of dimension r, and if these vector space structures and the restriction $\pi|_{\mathscr{F}}$ give $\mathscr{F}$ the structure of a vector bundle of rank r.
 (a) Let $\mathscr{L}$ be a vector bundle and $\mathscr{F}$ a subbundle of rank r. Let L be the typical fiber of $\mathscr{L}$. Fix a subspace $F \subset L$ of dimension r. Show that for each $m \in M$ there is a local trivialization of $\phi : \mathscr{L}|_U \to U \times L$ over a neighborhood U of m with the property that $\phi(\mathscr{F}|_U) = U \times F$. That is, a vector bundle and a subbundle can be *simultaneously locally trivialized*.
 (b) Let $\mathscr{L}'$ and $\mathscr{L}$ be vector bundles over M, and let $\phi : \mathscr{L} \to \mathscr{L}'$ be a bundle morphism over the identity of M. Show that the set of fiberwise kernels of ϕ (respectively, fiberwise images) form a subbundle of $\mathscr{L}$ (resp. $\mathscr{L}'$) if and only if these subspaces have the same dimension over each point of M.

(22) (This problem uses the concept of a subbundle, which is defined in exercise 21.) Let $\mathscr{L}$ be a vector bundle of rank n and $\mathscr{F}$ a subbundle of rank r. Show that the union of the fiberwise quotient vector spaces

$$\bigcup_{m \in M} \mathscr{L}_x / \mathscr{F}_x$$

has the structure of a vector bundle of rank $n - r$.

(23) (a) Calculate $H^k_\nabla(S^1, \mathscr{L})$ for an arbitrary connection ∇ on $\mathscr{L} = S^1 \times \mathbb{R}$. Describe an explicit basis for each space. Note that in this case the connection can be written as $D+\alpha$ for $\alpha \in \Omega^1(S^1)$; express your answer in terms of α.

(b) Repeat the same for the Möbius strip, considered as a line bundle over S^1. (First obtain a global description of an arbitrary connection on this bundle, analogous to the one given in the previous part for connections on $S^1 \times \mathbb{R}$.)

(24) Give an explicit formula for the operator d^k_∇ whose existence is the subject of Proposition 4.1.4 in the text.

(25) Recall that both the trivial line bundle $\mathscr{L}_0 := S^1 \times \mathbb{R}$ and the Möbius strip $\mathscr{L}_1$ can be viewed as quotients of the trivial bundle $\mathbb{R} \times \mathbb{R}$ by an action of $\mathbb{Z}$ by bundle automorphisms, with the generator acting by $(x,t) \mapsto (x+1, \pm t)$ (respectively). In each case, any periodic 1-form on $\mathbb{R}$ defines a flat connection on the quotient bundle. Denote the connection associated to $a\,dx$ by ∇_a (where $a \in \mathbb{R}$).

On the torus $T^2 = S^1 \times S^1$, let Π_1 and Π_2 denote the projections onto the two factors. For $i, j \in \{0,1\}$, let $\mathscr{L}_{ij}$ denote the line bundle $\Pi_1^*(\mathscr{L}_i) \otimes \Pi_2^*(\mathscr{L}_j)$. Thus for example $\mathscr{L}_{00}$ is the trivial line bundle, while $\mathscr{L}_{01}$ is nontrivial but restricts to a trivial bundle over each circle $S^1 \times \{pt\}$.

Now introduce a 2-parameter family of connections on $\nabla_{a,b}$ on $\mathscr{L}_{ij}$ defined by

$$\nabla_{a,b} = \Pi_1^*(\nabla_a) \otimes \Pi_2^*(\nabla_b).$$

Note that these are flat.

(a) Select a filling ribbon graph Γ in T^2 and compute the ribbon complex $C^\bullet_\Gamma$ and its cohomology (as a function of i, j, a, b). In particular, give bases for the cochain groups and matrices for the differentials.

(b) Do the same for the dual ribbon graph Γ^*.

(c) The dual of the flat bundle $(\mathscr{L}_{ij}, \nabla_{a,b})$ can be written in the same form, for different values of i, j, a, b. Determine these.

(d) Combine the previous parts to compute the full symplectic complex $C^\bullet$ associated to the pair of dual ribbon graphs Γ, Γ^*. That is, in addition to giving bases and differential matrices, also write matrices for the forms Ω_{02} and Ω_{11}.