4.2.1 The combinatorial complex

In order to smoothe our exposition, we have to consider the set of Γ_0 of oriented vertices, as well as the set Γ_2 of oriented faces, even though in our context vertices and faces are canonically oriented. We denote as usual $\overline{\alpha}$ the element α of Γ_i with the opposite orientation. The boundary $\partial\alpha$ of an oriented element α of Γ_i is a tuple of elements of Γ_{i-1}, possibly with repetition. For instance, if e is an edge then

$$\partial e = (e_+, \overline{e_-}).$$

A covering

We now consider for every $\alpha \in \Gamma_i$ a contractible open set U_α which is a neighbourhood of the interior $\hat{\alpha}$, that is a vertex, the interior of the edge and the face. We denote by W_i the union of all the open sets U_α so that α is in Γ_i. Finally we choose for every pair (α, β) so that $\alpha \in \partial\beta$, open sets $U_{\alpha,\beta}$ homeomorphic to disks.

- $U_{\overline{\alpha}} = U_\alpha$ and $U_{\overline{e},\overline{f}} = U_{e,f}$.
- $\forall i,\ \forall \alpha, \beta \in \Gamma_i, \text{with } \alpha \notin \{\beta, \overline{\beta}\},\quad \overline{U_\alpha} \cap \overline{U_\beta} = \emptyset.$
- For every edge e, $U_e \cap W_0 = U_{e^+,e} \sqcup U_{\overline{e}_-,e}$,
- For face f, $U_f \cap W_1 = \bigsqcup_{e \in \partial f} U_{e,f}$.

Vector spaces and homomorphisms

We now are given a vector bundle $\mathcal{L}$ equipped with a flat connection ∇.

We consider the vector space L_α, which consists of section parallel of $\mathcal{L}|_{U_\alpha}$. Observe that we have a canonical trivialisation of $\mathcal{L}\big|_{U_\alpha}$ as $L_\alpha \times U_\alpha$, and that $L_\alpha = L_{\overline{\alpha}}$.

Moreover, observe that for any pair (α, β) so that $\alpha \in \partial\beta$, there is a natural isomorphism $i_{\alpha,\beta}$ from L_α to L_β : if u is parallel section along U_α, $i_{\alpha,\beta}u$ is the unique parallel section along U_β which coincides with u on $U_{\alpha,\beta}$.

Exercise 4.2.1 Describe $i_{\alpha,\beta}$ using a trivialisation of the bundle at every vertex and the combinatorial connection associated to ∇.

A combinatorial complex

We consider the complex defined by the vector spaces

$$C^i_\Gamma = \{c^i : \Gamma_i \to \sqcup_{\alpha \in \Gamma_i} L_\alpha \ |c^i(\alpha) \in L_\alpha \text{ and } c^i(\overline{\alpha}) = -c^i(\alpha)\},$$

and the coboundary operators d by

$$\mathrm{d}_i : C^i_\Gamma \to C^{i+1}_\Gamma, \quad \mathrm{d}_i c^i(\beta_{i+1}) = \sum_{\alpha_i \in \partial\beta_{i+1}} i_{\alpha_i,\beta_{i+1}} c^i(\alpha_i).$$

One checks that $\mathrm{d} \circ \mathrm{d} = 0$. We define

$$H^i_\Gamma(L) = \mathrm{Ker}(\mathrm{d}_i)/\mathrm{Im}(\mathrm{d}_{i-1}).$$

4.2.2 The Isomorphism Theorem

In this section, we prove that the two versions of the cohomology that we have built are the same.

First we need to build a map between complexes. We associate to an $\omega \in \Omega^i(S; L)$ the element $\widehat{\omega}$ in C^i defined by

$$\widehat{\omega}(\alpha^i) = \int_{\alpha^i} \omega.$$

The integration is understood in the canonical trivialisation of $\mathcal{L}\big|_{U_{\alpha_i}}$ as $L_{\alpha_i} \times U_{\alpha_i}$, since we have an identification $\Omega^i(U_{\alpha_i}; \mathcal{L}) = \Omega^i(U_{\alpha_i}) \otimes L_{\alpha_i}$. We now claim

Proposition 4.2.2

$$\widehat{\mathrm{d}\omega} = \mathrm{d}\widehat{\omega}.$$

PROOF: This is an easy consequence of Stokes's Formula and we shall only check it when $i = 1$. We explain the technical details that we shall omit in the sequel. Let f be an element of Γ_2. Let $\partial f = \{e_1, \ldots, e_n\}$. We consider f as a map from the closed disk $\mathbf{D}$ to S. We observe that we can write $\partial\mathbf{D}$ as a reunion of closed intervals I_i so that $f\big|_{I_i}$ is a parametrisation of the edge e_i. By construction, the induced bundle $f^*\mathcal{L}$ is trivialised as $L_f \times \mathbf{D}$. As a consequence, if $\omega \in \Omega^1(S, L)$, then $f^*\omega \in \Omega(\mathbf{D}) \otimes L_f$. Now

$$\begin{aligned}\widehat{\mathrm{d}\omega}(f) &= \int_f \mathrm{d}\omega = \int_{\mathbf{D}} \mathrm{d}f^*\omega \\ &= \int_{\partial\mathbf{D}} f^*\omega \\ &= \sum_{i=1}^{i=n} \int_{I_i} f^*\omega.\end{aligned}$$

Finally, we remark that

$$\int_{I_i} f^*\omega = i_{e_i,f} \int_{e_i} \omega.$$

Hence

$$\widehat{\mathrm{d}\omega}(f) = \mathrm{d}\widehat{\omega}(f).$$

Q.E.D.

It follows from this identification that we have a natural map $u \mapsto \widehat{u}$ from $H^i_\nabla(S, \mathcal{L})$ to $H^i_\Gamma(S, \mathcal{L})$ so that

$$[\widehat{\omega}] = \widehat{[\omega]}$$

We now prove

Theorem 4.2.3 [Isomorphism theorem] *The map $u \mapsto \widehat{u}$ from $H^i_\nabla(S, \mathcal{L})$ to $H^i_\Gamma(S, \mathcal{L})$ is an isomorphism.*

Again, to shorten our exposition we only prove this result for $i = 1$. We prove this in two steps: injectivity and surjectivity of this map

Proposition 4.2.4 *The map $u \mapsto \widehat{u}$ from $H^i_\nabla(S, \mathcal{L})$ to $H^i_\Gamma(S, \mathcal{L})$ is surjective.*

Proof: We first prove that given $c^1 \in C^1$, there exists a neighbourhood U_1 of Γ with $U_1 \cap U_f$ is an annulus for all f, and a 1-form $\omega \in \Omega^1(S, L)$ so that

$$\begin{aligned} \widehat{\omega} &= c^1 \\ \mathrm{d}^\nabla \omega\big|_{U_1} &= 0. \end{aligned} \tag{4.3}$$

By linearity, it suffices to show this for c^1 such that there exists an edge e so that $c^1(\alpha) = 0$ if $\alpha \neq e$.

Let now φ be a real valued function defined on U_e so that $\varphi = 0$ on a neighbourhood of $U_{e_+,e}$ and $\varphi = 1$ on a neighbourhood of $U_{\overline{e_-},e}$. We now consider

$$\sigma = \varphi \cdot c^1(e) \in \Omega^0(U_e; \mathcal{L}).$$

Observe that $\mathrm{d}^\nabla \sigma = 0$ on $U_{\overline{e_-},e} \sqcup U_{e_+,e}$. It follows that $\beta = \mathrm{d}^\nabla \sigma$ can be extended smoothly to $W_1 \cup W_0$ by zero outside U_e. Let ψ is a function with support in $W_1 \cup W_0$ which is equal to 1 on a neighbourhood U_1 of Γ. Let

$$\omega = \psi\beta,$$

extended by 0 outside $W^1 \cup W_0$. Then ω fulfils our conditions (4.3).

Finally, let c^1, U_1 and ω as in Equations (4.3), and let's suppose that $\mathrm{d}c^1 = 0$. For any face f, Let γ_f be a circle which is a retract of the annulus $U_f \cap U_1$. We then have

$$\int_{\gamma_f} \omega = \mathrm{d}c^1(f) = 0.$$

It follows that $\omega|_{U_f\cap U_1} = \mathrm{d}\beta_f$. We extend β_f to U_f in any reasonnable smooth way and replace ω by $\mathrm{d}^\nabla\beta_f$ on U_f, in order to promote ω to a closed form on U_f. Performing this operation for every face f, we end up with a closed form ω so that $\widehat{\omega} = c^1$. Hence $u \mapsto \widehat{u}$ is indeed surjective. Q.E.D.

Proposition 4.2.5 *The map $u \mapsto \hat{u}$ from $H^i_\nabla(S,\mathcal{L})$ to $H^i_\Gamma(S,\mathcal{L})$ is injective.*

PROOF: We prove it only for $i = 1$ again. Let us assume that ω is closed and such that $\widehat{\omega} = \mathrm{d}c^0$. We wish to prove that ω is exact. We proceed by steps again.

We first notice that we can as well assume that $\omega = 0$ on W_0. Indeed ω – being closed – is exact on a neighbourhood U_0 of W_0:

$$\omega\Big|_{U_0} = \mathrm{d}\alpha.$$

Hence using a function φ with support in U_0 and equal to 1 on W_0 we replace ω by the cohomologous form

$$\omega - \mathrm{d}^\nabla(\varphi\alpha),$$

which satisfies $\omega|_{W_0} = 0$.

Now we show that we can as well assume that $\widehat{\omega} = 0$. Indeed, we choose a parallel section σ on U_0 so that for every vertex v, we have $\sigma(v) = c^0(v)$. We here choose U_0 to have one connected component by vertex. It follows that $\omega - \mathrm{d}^\nabla\varphi\sigma$ satisfies the required condition.

Next we show that we can reduce to the case that $\omega = 0$ on a neighbourhood of Γ. Indeed, for every edge e, since ω is closed, $\omega|_{U_e} = \mathrm{d}\alpha_e$, where α is and defined on U_e. By construction α_e is now parallel on $U_0 \cap U_e$. We can choose α_e so that $\alpha_e = 0$ on $U_{e+,e}$: indeed α_e is parrallel on a neighbourhood O of $\overline{U}_{e^+,e}$. Thus $\alpha_e = \mathrm{d}^\nabla\beta$ on such a neighbourhood. We may now replace α_e by $\alpha_e - d^\nabla\varphi\beta$, where φ has support in O and is equal to 1 on $U_{e+,e}$. Since $\int_e \omega = \widehat{\omega}(e) = 0$ it follows that α_e is zero on the other connected component of $U_0 \cap U_e$. Therefore, smoothing again by a function ψ with support in W_1 and equal to 1 on a neighbourhood of Γ, we get that

$$\omega - \mathrm{d}\left(\sum_{e\in E}\alpha_e\right),$$

is zero on a neighbourhood of Γ.

Finally, we observe that for every face f, $\omega|_f = \mathrm{d}\beta_f$. By our condition β_f is parrallel (and constant in the trivialisation) on a neighbourhood of Γ; therefore, substracting this constant, we can choose β_f to be zero on a (connected) neighbourhood of Γ. In particular $\beta = \sum_{f\in F} \beta_f$ makes sense and we have

$$\omega = \mathrm{d}\beta.$$

This concludes the proof.

4.2.3 Duality

In this section, we give the two versions of Poincaré duality: one for the de Rham cohomology, one for the combinatorial version.

Symplectic complexes

Definition 4.2.6 *We say a complex $C^\bullet : 0 \to C^0 \stackrel{\mathrm{d}}{\to} C^1 \stackrel{\mathrm{d}}{\to} C^2$ of degree 2 is* symplectic*, if we have a*

- *a symplectic form ω on C^1,*
- *a non-degenerate pairing ω on $C^0 \times C^2$ such that*

$$\omega(\mathrm{d}\alpha_0, \alpha_1) = \omega(\alpha_0, \mathrm{d}\alpha_1).$$

Proposition 4.2.7 *The first cohomology is a symplectic vector space with symplectic form $[\omega]$ such that*

$$[\omega]([\alpha],[\beta]) = \omega(\alpha, \beta).$$

Proof: this follows at once from the fact that

$$\ker(\mathrm{d}_1)^o = \mathrm{im}(\mathrm{d}_0), \tag{4.4}$$

where V^o denote the orthogonal with respect to ω of $V \subset C^1$. Q.E.D.

The dual graph

We realise geometrically the dual graph Γ^* in S. We denote by $\alpha \mapsto \alpha^*$ the map from Γ_i to γ_{2-i}. As far as the boundary is concerned, we observe that

$$v \in \partial e \implies \overline{e^*} \in \partial v^*, \tag{4.5}$$
$$e \in \partial f \implies f^* \in \partial e^*. \tag{4.6}$$

If $\alpha \in \Gamma_0^*$, we choose a neighbourhood U_α^* as above, requiring furthermore that for $v \in \Gamma^0$, $U_v \subset U_{v^*}^*$, for $e \in \Gamma^1$, $U_e \cap U_{e^*}^*$ is contractible, and for