BERS SLICES ARE ZARISKI DENSE

DAVID DUMAS AND RICHARD P. KENT IV

Let S be a closed oriented surface of genus $g \geq 2$, and let $\mathcal{X}_{\mathbb{C}}(S)$ denote the variety of $\mathrm{SL}_2\,\mathbb{C}$-characters of $\pi_1(S)$. The quasi-Fuchsian space $\mathcal{QF}(S)$ is an open subset of the smooth locus of $\mathcal{X}_{\mathbb{C}}(S)$ equipped with a biholomorphic parametrization by a product of Teichmüller spaces

$$Q : \mathcal{T}(S) \times \mathcal{T}(\overline{S}) \to \mathcal{QF}(S).$$

The space $\mathcal{QF}(S)$ parametrizes the quasi-Fuchsian representations of $\pi_1(S)$, and the parametrization Q is due to Bers [Ber2].

Picking a point in one factor gives a *Bers slice*

$$\mathcal{B}_Y = Q\left(\mathcal{T}(S) \times \{Y\}\right) \subset \mathcal{QF}(S).$$

Each Bers slice is a holomorphically embedded copy of Teichmüller space within $\mathcal{X}_{\mathbb{C}}(S)$. While it follows that $\mathcal{B}_Y$ can be locally described as the common zero locus of finitely many analytic functions on $\mathcal{X}_{\mathbb{C}}(S)$, it is known that the Bers slice is *not* a locally algebraic set [DK]—this is used to show that W. Thurston's skinning map is not a constant function [DK].

We prove a stronger result about the transcendence of $\mathcal{B}_Y$:

Theorem A. *Let S be a closed surface of genus $g \geq 2$. For any $Y \in \mathcal{T}(S)$, the Bers slice $\mathcal{B}_Y$ is Zariski dense in $\mathcal{X}_{\mathbb{C}}(S)$.*

The proof is roughly as follows: The Zariski closure of the Bers slice $\mathcal{B}_Y$ contains an analytic subvariety $\mathcal{W}_Y \subset \mathcal{X}_{\mathbb{C}}(S)$ of complex dimension $-\frac{3}{2}\chi(S)$ which consists of holonomy representations of projective structures on Y. By analyzing its real points, we show that $\mathcal{W}_Y$ accumulates on a $(-3\chi(S)-1)$–dimensional subset of the logarithmic limit set of $\mathcal{X}_{\mathbb{C}}(S)$, whereas a proper subvariety of $\mathcal{X}_{\mathbb{C}}(S)$ accumulates on a set of strictly smaller dimension. Thus $\mathcal{W}_Y$, and hence $\mathcal{B}_Y$, is Zariski dense.

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1. Logarithmic Limit Sets

For an affine algebraic variety $V \subset \mathbb{C}^n$, let $\operatorname{Log}|V| \subset \mathbb{R}^n$ denote the *amoeba* of V: the set of points $\{(\log|z_1|, \ldots, \log|z_n|) \mid z \in V, z_i \neq 0\}$.

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The *logarithmic limit set* of V, denoted V^∞, is the set of limit points of $\operatorname{Log}|V|$ on the positive projectivization $S^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+$. Equivalently, considering S^{n-1} as the boundary of the unit ball B^n, we have

$$V^\infty = \overline{f(\operatorname{Log}|V|)} \cap S^{n-1},$$

where $f : \mathbb{R}^n \to B^n$ is the rescaling map $f(x) = \frac{x}{1+\|x\|}$.

From this description, we see that V^∞ is the boundary of a *logarithmic compactification* of $V^* = V \cap (\mathbb{C}^*)^n$, which we denote by $\overline{V} = V^* \sqcup V^\infty$. Explicitly, a sequence $z^{(i)} \in V^*$ converges to the ray $[p] = \mathbb{R}^+ \cdot p \in S^{n-1}$, where $p \in \mathbb{R}^n - \{0\}$, if there exists a sequence $c_i \in \mathbb{R}^+$ such that $c_i \to 0$ and

$$\lim_{i\to\infty} \left(c_i \log|z_1^{(i)}|, \ldots, c_i \log|z_n^{(i)}| \right) = (p_1, \ldots, p_n). \tag{1.1}$$

Given a subset $E \subset S^{n-1}$, let $\operatorname{Cone}(E) = (\mathbb{R}^+ \cdot E) \cup \{0\} \subset \mathbb{R}^n$ denote the *cone* on E. Note that $E = \operatorname{Cone}(E) \cap S^{n-1}$. Some properties of the logarithmic limit set V^∞ are most easily expressed in terms of $\operatorname{Cone}(V^\infty)$. For example, G. Bergman proved the fundamental structure theorem:

Theorem 1.1 ([Ber1]). *If $V \subset \mathbb{C}^n$ is an affine algebraic variety of complex dimension d, then* $\operatorname{Cone}(V^\infty)$ *is contained in a finite union of d-dimensional subspaces of $\mathbb{R}^n$ defined over $\mathbb{Q}$.*

Equivalently, the logarithmic limit set V^∞ is contained in a finite union of *rational great $(d-1)$-spheres*.

Strengthening the theorem above, Bieri and Groves [BG] showed that $\operatorname{Cone}(V^\infty)$ is a finite union of rational polyhedral convex cones—sets of the form $\{\sum_{i=1}^k c_i x_i \,|\, c_i \geq 0\}$ where $x_i \in \mathbb{Q}^n$—and that at least one of these cones spans a subspace of dimension d. However, for our purposes, the dimension estimate of Theorem 1.1 will suffice.

We extend the definition of logarithmic limit sets to apply to subsets of algebraic varieties: for $T \subset V$, we denote by T^∞ the boundary of $T \cap V^*$ in $\overline{V}$. In our cases of interest, T will be a noncompact and properly embedded submanifold of the smooth points of V^*, and in particular T^∞ will be nonempty.

2. Character varieties

Let $\mathcal{X}_{\mathbb{C}}(S) = \operatorname{Hom}(\pi_1(S), \operatorname{SL}_2\mathbb{C}) /\!\!/ \operatorname{SL}_2\mathbb{C}$ denote the $\operatorname{SL}_2\mathbb{C}$ character variety of the compact surface S, which is an irreducible affine variety of dimension $6g-6$ (see [CS] [Gol2]). For a suitable finite subset $\{\gamma_1, \ldots, \gamma_N\}$ of $\pi_1(S)$, the variety $\mathcal{X}_{\mathbb{C}}(S)$ can be realized as the image of $\operatorname{Hom}(\pi_1(S), \operatorname{SL}_2\mathbb{C})$ by the trace map

$$\rho \mapsto (\operatorname{tr}\rho(\gamma_1), \ldots, \operatorname{tr}\rho(\gamma_N)).$$

We call such $\{\gamma_i\}$ an *embedding set for* $\mathcal{X}_{\mathbb{C}}(S)$. For example, given a generating set for $\pi_1(S)$ with n elements, the set of words of length at most n in these generators is an embedding set for $\mathcal{X}_{\mathbb{C}}(S)$ [CS, §1.4]. The traces of the images of an embedding set determine the traces of all elements of $\rho(\pi_1(S))$—that is, they determine the *character* of ρ.

We will only consider the characters of nonelementary representations $\rho : \pi_1(S) \to \mathrm{SL}_2\,\mathbb{C}$, which are in one-to-one correspondence with conjugacy classes of such representations. When convenient, we blur the distinction between a character and representatives of the associated conjugacy class of representations.

Each marked hyperbolic structure on S has a corresponding Fuchsian representation $\rho : \pi_1(S) \to \mathrm{SL}_2\,\mathbb{R}$ (one of the finitely many lifts of $\rho_0 : \pi_1(S) \to \mathrm{PSL}_2\,\mathbb{R} \simeq \mathrm{Isom}^+(\mathbb{H}^2)$). In this way, the Teichmüller space $\mathcal{T}(S)$ can be identified with a connected component of the set of real points $\mathcal{X}_\mathbb{R}(S) \subset \mathcal{X}_\mathbb{C}(S)$ [Gol2]. Similarly, the quasi-Fuchsian space $\mathcal{QF}(S)$ of equivariant quasiconformal deformations of Fuchsian representations is identified with an open neighborhood of $\mathcal{T}(S)$ in $\mathcal{X}_\mathbb{C}(S)$ [Mar] [Sul].

3. Measured laminations and Thurston's compactification

Let $\mathcal{ML}(S)$ denote the space of measured geodesic laminations on S, which is a piecewise–linear manifold homeomorphic to $\mathbb{R}^{6g-6}$. Let $\mathcal{S}$ denote the set of isotopy classes of homotopically nontrivial simple closed curves on S, and note that each element of $\mathcal{S}$ corresponds to a conjugacy class $[\gamma] \subset \pi_1(S)$. Then $\mathbb{R}^+ \times \mathcal{S}$ is naturally a dense subset of $\mathcal{ML}(S)$ consisting of weighted simple closed geodesics.

For a suitable finite subset $\{\gamma_1, \ldots, \gamma_N\}$ of $\pi_1(S)$, we have a piecewise–linear embedding $\mathcal{ML}(S) \to \mathbb{R}^N$,

$$\lambda \mapsto (i(\lambda, \gamma_1), \ldots, i(\lambda, \gamma_N))$$

where $i(\lambda, \gamma)$ is the *intersection number*, the minimum mass assigned to a representative of the isotopy class of γ by the transverse measure of $\lambda \in \mathcal{ML}(S)$. (The intersection number for simple curves is discussed in [FLP, Ch. 5]; for closed but possibly self-intersecting curves, see [Bon].) We call such $\{\gamma_1, \ldots \gamma_N\} \subset \pi_1(S)$ an *embedding set for* $\mathcal{ML}(S)$. An example of an embedding set with $9g - 9$ elements is described in [FLP, §6.4]. The image of such an embedding is a piecewise–linear cone, and the space $\mathbb{P}\mathcal{ML}(S)$ of rays in $\mathcal{ML}(S)$ can be identified with $\mathcal{ML}(S) \cap S^{N-1}$.

Thurston's compactification of Teichmüller space adjoins $\mathbb{P}\mathcal{ML}(S)$ as the boundary of $\mathcal{T}(S)$ according to the asymptotic behavior of hyperbolic lengths of geodesics. Specifically, a sequence $X_n \in \mathcal{T}(S)$ converges to $[\lambda] = \mathbb{R}^+\lambda \in \mathbb{P}\mathcal{ML}(S)$ if and only if there is a sequence $c_n \in \mathbb{R}^+$ such that $c_n \to 0$ and

$$\lim_{n\to\infty} c_n \ell(\alpha, X_n) \to i(\lambda, \alpha) \tag{3.1}$$

for all closed curves α. Thurston showed that the same compactification is obtained if (3.1) is required only for the finitely many α in an embedding family for $\mathcal{ML}(S)$. For details, see [FLP, Ch. 8] [Bon, Thm. 18].

4. Logarithmic limit sets and Thurston's compactification

We fix throughout a finite set $\{\gamma_1, \ldots, \gamma_N\} \subset \pi_1(S)$ that is both an embedding set for $\mathcal{X}_{\mathbb{C}}(S)$ and for $\mathcal{ML}(S)$ (e.g. the union of the example embedding sets described above). We use this family to identify $\mathcal{X}_{\mathbb{C}}(S)$ and $\mathcal{ML}(S)$ with their embedded images in $\mathbb{C}^N$ and $\mathbb{R}^N$, respectively. Similarly, we regard $\mathbb{P}\mathcal{ML}(S)$ as a subset of S^{N-1}.

With this embedding of $\mathcal{X}_{\mathbb{C}}(S)$, we can consider the logarithmic limit sets $\mathcal{X}_{\mathbb{C}}(S)^\infty$ and $\mathcal{T}(S)^\infty$ in S^{N-1}.

Lemma 4.1. *We have $\mathcal{T}(S)^\infty = \mathbb{P}\mathcal{ML}(S)$, and the logarithmic compactification of $\mathcal{T}(S)$ is identical to the Thurston compactification.*

Proof. If $\rho_X \in \mathrm{Hom}(\pi_1(S), \mathrm{SL}_2 \mathbb{R})$ represents $X \in \mathcal{T}(S)$, then its character is real and satisfies $|\operatorname{tr} \rho(\gamma)| \geq 2$ for all γ, with equality only when $\gamma = 1$. The hyperbolic length and trace of $\alpha \in \mathcal{S}$ are related by

$$\ell(\alpha, X) = 2 \operatorname{arccosh} \frac{|\operatorname{tr} \rho(\alpha)|}{2}.$$

Since $\operatorname{arccosh} \frac{|t|}{2} = \log |t| + o(1)$ as $t \to \infty$, then for any $X_i \in \mathcal{T}(S)$ and $c_i \to 0$, the sequences $c_i \operatorname{arccosh} \frac{|\operatorname{tr} \rho_{X_i}(\gamma)|}{2}$ and $c_i \ell(\gamma, X_i)$ have the same limit, and one converges if and only if the other does. Applying this to the $\gamma_1, \ldots, \gamma_N$ demonstrates the equivalence of convergence in the Thurston compactification (3.1) and convergence in the logarithmic one (1.1). $\square$

Remark. The lemma above is essentially the same as Theorem III.3.2 of [MS], which identifies the Morgan–Shalen and Thurston compactifications of Teichmüller space. The definition of the Morgan–Shalen compactification is similar to that of the logarithmic compactification, except that it is defined using the function $\log(2 + |\operatorname{tr} \rho(\cdot)|)$ rather than $\log |\operatorname{tr} \rho(\cdot)|$; and by considering the limiting behavior of the traces of *all* elements of $\pi_1(S)$, rather than a finite subset. The difference in the logarithmic scaling function does not affect projective limits, and since finitely many intersection number functions embed $\mathcal{ML}(S)$ into $\mathbb{R}^n$, Lemma 4.1 follows. Morgan and Shalen note that their Theorem III.3.2 is, in turn, equivalent to results of Thurston. The connection between Thurston's compactification and logarithmic limit sets is explained in detail in recent papers of Alessandrini ([Ale2] and [Ale1, §6]), to which we refer the reader for a complete discussion.

5. Quadratic differentials and holonomy

For any $Y \in \mathcal{T}(S)$, let $Q(Y) \simeq \mathbb{C}^{3g-3}$ denote the vector space of holomorphic quadratic differentials on the Riemann surface Y. We identify this vector space with the set of *complex projective structures* on Y: conformal atlases whose transition functions are Möbius maps. Under this identification, the zero quadratic differential corresponds to the Fuchsian uniformization $\widetilde{Y} \simeq \mathbb{H}$, and, more generally, $\phi \in Q(Y)$ corresponds to a projective structure whose developing map $f_\phi : \mathbb{H} \to \mathbb{CP}^1$ has Schwarzian derivative $S(f_\phi) = \phi$.

The *holonomy map* is the holomorphic map

$$\mathrm{hol} : Q(Y) \to \mathcal{X}_{\mathbb{C}}(S)$$

which sends a quadratic differential ϕ to the conjugacy class of the holonomy representation of the associated projective structure on Y. Here we implicitly lift these holonomy representations from $\mathrm{PSL}_2\,\mathbb{C}$ to $\mathrm{SL}_2\,\mathbb{C}$, which requires the choice of a spin structure on Y, or equivalently, a choice of cohomology class in $H^1(Y, \pi_1(\mathrm{PSL}_2\,\mathbb{C}))$. The particular choice among the finite set of spin structures will not concern us.

The map hol is a proper holomorphic embedding, so its image

$$\mathcal{W}_Y = \mathrm{hol}(Q(Y)) \subset \mathcal{X}_{\mathbb{C}}(S)$$

is an analytic subvariety [GKM, §11.4] (also see [D2, §5.7] for the history of this theorem). We also have $\mathcal{B}_Y \subset \mathcal{W}_Y$, and in fact, an open set $B \subset Q(Y)$ (the *Bers embedding of* $\mathcal{T}(Y)$) maps biholomorphically to $\mathcal{B}_Y$ under hol. This leads to a key observation used in the proof of Theorem A:

Lemma 5.1 ([DK])**.** *Any analytic subvariety of* $\mathcal{X}_{\mathbb{C}}(S)$ *containing* $\mathcal{B}_Y$ *also contains* $\mathcal{W}_Y$*. In particular, the Zariski closure of* $\mathcal{B}_Y$ *contains* $\mathcal{W}_Y$*.*

Thus to prove the main theorem, it suffices to show that $\mathcal{W}_Y$ is Zariski dense in $\mathcal{X}_{\mathbb{C}}(S)$.

6. Grafting

We now describe certain points on $\mathcal{W}_Y$ in terms of grafting.

Equip $X \in \mathcal{T}(S)$ with its hyperbolic metric, and let γ be a simple closed geodesic on X. Removing γ from X and replacing it with a Euclidean cylinder $\gamma \times [0, t]$, we obtain a new surface with a well-defined conformal structure. This is the *grafting of* X *by* $t\gamma$, denoted $\mathrm{gr}_{t\gamma} X$. By adjoining multiple cylinders, grafting extends naturally to measured laminations $\lambda = \sum_i c_i \gamma_i$ supported on unions of disjoint simple closed geodesics. The case when $c_i \in 2\pi\mathbb{Z}$ will be of particular interest to us, and we let $2\pi\mathcal{ML}_{\mathbb{Z}}(S)$ be the set of all such 2π-integral measured laminations.

Theorem 6.1 (Tanigawa, [Tan])**.** *For each* $\lambda \in 2\pi\mathcal{ML}_{\mathbb{Z}}(S)$*, the map* $\mathrm{gr}_\lambda : \mathcal{T}(S) \to \mathcal{T}(S)$ *is a diffeomorphism.*

Remark. Scannell and Wolf have shown that the same result holds for all $\lambda \in \mathcal{ML}(S)$ [SW].

So we have an inverse map $\mathrm{gr}_\lambda^{-1} : \mathcal{T}(S) \to \mathcal{T}(S)$. Goldman showed that grafting can be used to describe the intersection $\mathcal{W}_Y \cap \mathcal{T}(S)$ explicitly:

Theorem 6.2 (Goldman, [Gol1])**.** *For each* Y *we have*

$$\mathcal{W}_Y \cap \mathcal{T}(S) = \{\mathrm{gr}_\lambda^{-1}(Y) \mid \lambda \in 2\pi\mathcal{ML}_{\mathbb{Z}}(S)\}.$$

The map $\lambda \mapsto \mathrm{gr}_\lambda^{-1}(Y)$ is injective (see [DW]), establishing a bijection between $\mathcal{W}_Y \cap \mathcal{T}(S)$ and $\mathcal{ML}_{\mathbb{Z}}(S)$. In particular, the set $\mathcal{W}_Y \cap \mathcal{T}(S)$ is infinite, which is used in [DK] to show that $\mathcal{W}_Y$ is not an algebraic variety.

7. Antipodal limits

Each quadratic differential $\phi \in Q(Y)$ defines a pair of orthogonal singular foliations of the Riemann surface Y, the *horizontal* and *vertical* foliations, whose leaves integrate the distributions $\{v \in TY \mid \phi(v) \geq 0\}$ and $\{v \in TY \mid \phi(v) \leq 0\}$, respectively. Straightening the leaves of the horizontal foliation with respect to the hyperbolic metric yields a measured lamination $\Lambda_Y(\phi) \in \mathcal{ML}(S)$ (see [Lev]), and Hubbard and Masur showed that $\Lambda_Y : Q(Y) \to \mathcal{ML}(S)$ is a homeomorphism [HM]. Similarly, the map $\phi \mapsto \Lambda_Y(-\phi)$ corresponds to straightening the vertical foliation, and is also a homeomorphism.

Define the Y*-antipodal map* $i_Y : \mathcal{ML}(S) \to \mathcal{ML}(S)$ by

$$i_Y(\lambda) = \Lambda_Y(-\Lambda_Y^{-1}(\lambda)).$$

Thus i_Y is an involutive homeomorphism that exchanges the laminations corresponding to vertical and horizontal foliations of quadratic differentials on Y. It is easy to see that for all $c > 0$, we have $i_Y(c\lambda) = c i_Y(\lambda)$, and thus i_Y descends to a homeomorphism $\mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$, which we also call i_Y.

We need the following result from [D1].

Theorem 7.1. *Let $\lambda_n \to \infty$ be a divergent sequence in $\mathcal{ML}(S)$ such that the rays $[\lambda_n] \in \mathbb{P}\mathcal{ML}(S)$ converge to $[\lambda]$. Then*

$$\lim_{n\to\infty} \mathrm{gr}_{\lambda_n}^{-1}(Y) = [i_Y(\lambda)]$$

in the Thurston compactification.

Remark. While we have stated the above Theorem for arbitrary sequences of laminations, for our purposes it is enough to consider limits of sequences $\mathrm{gr}_{2\pi n\gamma}^{-1}(Y)$, where $\gamma \in \mathcal{S}$ and $n \to \infty$. The proof of Theorem 7.1 in this special case is somewhat simpler.

Combining this with the grafting description of $\mathcal{W}_Y \cap \mathcal{T}(S)$, we have:

Corollary 7.2. *For any $Y \in \mathcal{T}(S)$, the set $\mathcal{W}_Y \cap \mathcal{T}(S)$ accumulates on the entire Thurston boundary $\mathbb{P}\mathcal{ML}(S) = \mathcal{T}(S)^\infty$.*

Proof. By Theorems 7.1 and 6.2, the closure of $\mathcal{W}_Y \cap \mathcal{T}(S)$ contains all points $[\mu] \in \mathbb{P}\mathcal{ML}(S)$ of the form $i_Y([\lambda])$, where $\lambda \in 2\pi\mathcal{ML}_{\mathbb{Z}}(S)$. Since the rays $\{[\lambda] \,|\, \lambda \in 2\pi\mathcal{ML}_{\mathbb{Z}}(S)\}$ are dense in $\mathbb{P}\mathcal{ML}(S)$, and $i_Y : \mathbb{P}\mathcal{ML}(S) \to \mathbb{P}\mathcal{ML}(S)$ is a homeomorphism, we find that $\overline{\mathcal{W}_Y \cap \mathcal{T}(S)}$ is also dense in $\mathbb{P}\mathcal{ML}(S)$. Since $\overline{\mathcal{W}_Y \cap \mathcal{T}(S)}$ is closed, the Corollary follows. ∎

8. Density

We have seen in §5 that Theorem A follows from

Theorem 8.1. *The Zariski closure of $\mathcal{W}_Y$ is $\mathcal{X}_{\mathbb{C}}(S)$.*

Proof. Let $V \subset \mathcal{X}_{\mathbb{C}}(S)$ denote the Zariski closure of $\mathcal{W}_Y$. By Corollary 7.2 and Lemma 4.1, we have $\mathbb{P}\mathcal{ML}(S) \subset V^\infty$ and thus $\mathcal{ML}(S) \subset \mathrm{Cone}(V^\infty)$. In particular $\mathrm{Cone}(V^\infty)$ contains an open subset of a subspace of $\mathbb{R}^N$ of dimension $-3\chi(S)$.

On the other hand, by Theorem 1.1, we have that $\mathrm{Cone}(V^\infty)$ is contained in a finite union of subspaces of real dimension $d = \dim_{\mathbb{C}}(V)$. Thus $\dim_{\mathbb{C}} V \geq -3\chi(S) = \dim_{\mathbb{C}} \mathcal{X}_{\mathbb{C}}(S)$. Since the variety $\mathcal{X}_{\mathbb{C}}(S)$ is irreducible (see [Gol2]), the Theorem follows. □

The dimension count above also shows:

Corollary 8.2. *For any $Y \in \mathcal{T}(S)$, the set $\{\mathrm{gr}_\lambda^{-1}(Y) \mid \lambda \in 2\pi\mathcal{ML}_{\mathbb{Z}}(S)\}$ is Zariski dense in $\mathcal{X}_{\mathbb{R}}(S)$. The same is true of $\{\mathrm{gr}_\lambda^{-1}(Y) \mid \lambda \in E\}$ for any $E \subset 2\pi\mathcal{ML}_{\mathbb{Z}}(S)$ that accumulates on an open subset of $\mathbb{P}\mathcal{ML}(S)$.*

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Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago
ddumas@math.uic.edu

Department of Mathematics, Brown University
rkent@math.brown.edu