Geometry of compact complex manifolds associated to generalized quasi-Fuchsian representations

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1 Introduction

This paper is concerned with the following general question: which aspects of the complex-analytic study of discrete subgroups of $\mathrm{PSL}_2\mathbb{C}$ can be generalized to discrete subgroups of other semisimple complex Lie groups?

To make this more precise, we recall the classical situation that motivates our discussion. A torsion-free cocompact Fuchsian group $\Gamma < \mathrm{PSL}_2\mathbb{R}$ acts freely, properly discontinuously, and cocompactly by isometries on the symmetric space $\mathrm{PSL}_2\mathbb{R}/\mathrm{PSO}(2) \simeq \mathbb{H}^2$. The quotient $S = \Gamma\backslash\mathbb{H}^2$ is a closed surface of genus $g \geqslant 2$. When considering Γ as a subgroup of $\mathrm{PSL}_2\mathbb{C}$, it is natural to consider either its isometric action on the symmetric space $\mathbb{H}^3 \simeq \mathrm{PSL}_2\mathbb{C}/\mathrm{PSU}(2)$ or its holomorphic action on the visual boundary $\mathbb{P}^1_{\mathbb{C}} \simeq \mathrm{PSL}_2\mathbb{C}/B_{\mathrm{PSL}_2\mathbb{C}}$. The latter action has a limit set $\Lambda = \mathbb{P}^1_{\mathbb{R}}$ and a disconnected domain of discontinuity $\Omega = \mathbb{H} \sqcup \overline{\mathbb{H}}$. The quotient $\Gamma\backslash\Omega$ is a compact Kähler manifold—more concretely, it is the union of two complex conjugate Riemann surfaces.

Quasiconformal deformations of such groups Γ give *quasi-Fuchsian groups* in $\mathrm{PSL}_2\mathbb{C}$. Each such group acts on $\mathbb{P}^1_{\mathbb{C}}$ in topological conjugacy with a Fuchsian group, hence the limit set Λ is a Jordan curve, the domain of discontinuity has two contractible components, and the quotient manifold is a union of two Riemann surfaces (which are not necessarily complex conjugates of one another).

If G is a complex simple Lie group of adjoint type (such as $\mathrm{PSL}_n\mathbb{C}$, $n \geqslant 2$), there is a distinguished homomorphism $\iota_G : \mathrm{PSL}_2\mathbb{C} \to G$ introduced by Kostant [Kos59] and called the *principal three-dimensional embedding.* Applying ι_G, a discrete subgroup of $\mathrm{PSL}_2\mathbb{R}$ or $\mathrm{PSL}_2\mathbb{C}$ gives rise to a discrete subgroup of G. When this construction is applied to a torsion-free cocompact Fuchsian group $\pi_1 S \simeq \Gamma$, the resulting *G-Fuchsian representation* $\pi_1 S \to G$ lies in the Hitchin component of the split real form $G_{\mathbb{R}} < G$. Representations in the Hitchin component have been extensively studied in recent years, and the resulting rich geometric theory has shown them to be a natural higher-rank generalization of Fuchsian

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groups. In the same way, we propose to generalize the theory of quasi-Fuchsian groups by studying complex deformations of these G-Fuchsian and Hitchin representations and the associated holomorphic actions on parabolic homogeneous spaces of G.

The existence of domains of proper discontinuity for such actions follows from a theory developed by Kapovich-Leeb-Porti [KLP13] and Guichard-Wienhard [GW12], which applies in the more general setting of *Anosov representations* of word-hyperbolic groups in a semisimple[1] Lie group G. In fact, a key component of this theory, as developed in [KLP13], is the construction of many distinct domains of discontinuity (and of cocompact discontinuity) for the action of a given Anosov representation on a parabolic homogeneous space G/P, each labeled by a certain combinatorial object—*a Chevalley-Bruhat ideal* in the Weyl group of G.

Applying this theory to a G-Fuchsian representation of a surface group, or more generally to an Anosov representation of a word-hyperbolic group in a complex adjoint group G, we consider the compact, complex quotient manifold $\mathcal{W} = \Gamma \backslash \Omega$ associated to a cocompact domain of discontinuity $\Omega \subset G/P$ arising from the construction of [KLP13]. Concerning such a manifold, we ask:

- What is the homology of $\mathcal{W}$?
- Does $\mathcal{W}$ admit a Kähler metric? Is it a projective algebraic variety?
- What is the Picard group of $\mathcal{W}$?
- What are the cohomology groups of holomorphic line bundles on $\mathcal{W}$?
- Does $\mathcal{W}$ admit nonconstant meromorphic functions?

In considering these questions, our restriction to complex Lie groups has the simultaneous advantage that it simplifies topological questions, and that it paves the way for the rich holomorphic geometry of generalized flag varieties over $\mathbb{C}$ to assume a prominent role.

Our answers to these questions rest on the fact that if $\mathcal{W}$ were replaced by one of the complex partial flag varieties G/P, classical Lie theory would give a complete answer: The homology of G/P admits a preferred basis in terms of *Schubert cells*, which are B-orbits on G/P where $B < G$ is a Borel subgroup. The classification of line bundles on G/B and their sheaf cohomology is the content of the Borel-Bott-Weil theorem [Bot57].

In the remainder of this introduction we survey our results, after introducing enough terminology to formulate them precisely.

Choosing Cartan and Borel subgroups $H < B < G$ we obtain the Weyl group W and a natural partial order on it, the *Chevalley-Bruhat order*. A subset $I \subset W$ which is

[1]For this paper, a semisimple Lie group G is a real Lie group with finite center, finitely many connected components, semisimple Lie algebra, and with no compact factors. For the reader who prefers algebraic groups, one may also work with the $\mathbb{K}$-points of a semisimple linear algebraic group defined over $\mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on the situation.

downward-closed for this order is a *Chevalley-Bruhat ideal* (or briefly, an *ideal*). An ideal I is *balanced* if $W = I \sqcup w_0 I$ where $w_0 \in W$ is the unique element of maximal length.

Each element of W corresponds to a *Schubert cell* in the space G/B. The union of the cells corresponding to elements of an ideal I gives a closed set $\Phi^I \subset G/B$, the *model thickening*. For a general parabolic subgroup $P < G$, there is a similar construction of a model thickening $\Phi^I \subset G/P$ if we also assume that I is invariant under right multiplication by $W_P < W$, the Weyl group of the parabolic.

Now let π be a word hyperbolic group. A homomorphism $\varrho : \pi \to G$ is B-Anosov if there exists a ϱ-equivariant continuous map

$$\xi : \partial_\infty \pi \to G/B$$

which satisfies certain additional properties that are described in Section 2.5; roughly speaking, these conditions say that ϱ is "undistorted" at a large scale; in particular such a representation is a discrete, quasi-isometric embedding with finite kernel. The map ξ is the *limit curve* associated to the Anosov representation ϱ. (Section 2.5 also describes a more general notion of Anosov representation where B is replaced by an arbitrary symmetric parabolic subgroup of G.)

The work of Kapovich-Leeb-Porti [KLP13] establishes that if $\varrho : \pi \to G$ is B-Anosov, then for every balanced and right-W_P-invariant ideal $I \subset W$ one obtains a $\Gamma := \varrho(\pi)$-invariant open set $\Omega \subset G/P$ upon which the action of Γ is properly discontinuous and cocompact. The set Ω is defined as the complement $\Omega = (G/P) - \Lambda$ where Λ is a union over points in the limit curve ξ of G-translates of the model thickening Φ^I.

Using the continuous variation of the limit curve as a function of the Anosov representation (established in [GW12]), and the fact that the Anosov property is an open condition among representations (ibid.), it follows that if ϱ and ϱ' are in the same path component of the space of Anosov representations, then the corresponding compact quotient manifolds are homotopy equivalent. This homotopy equivalence can be upgraded to diffeomorphism provided we restrict to the smooth part of the space of representations.

We focus on the path component of the space of B-Anosov representations $\pi_1 S \to G$ that contains the G-Fuchsian representations, which we regard as a complex analogue of the Hitchin component of $G_{\mathbb{R}}$. This component also contains the compositions of quasi-Fuchsian representations with ι_G, which we call *G-quasi-Fuchsian representations*. Using the invariance of topological type described above, when studying topological invariants of quotient manifolds for representations in this component, it suffices to consider the G-Fuchsian case. Concerning homology, we find:

Theorem A. *Let G be a complex simple Lie group of adjoint type and let $\varrho : \pi_1 S \to G$ be a G-Fuchsian representation. Let $I \subset W$ be a balanced and right-W_P-invariant ideal, where $P < G$ is parabolic. Then if $\Omega_\varrho^I \subset G/P$ is the corresponding cocompact domain of discontinuity, the quotient manifold $\mathcal{W}_\varrho^I = \varrho(\pi_1 S) \backslash \Omega_\varrho^I$ satisfies*

$$H_*(\mathcal{W}_\varrho^I, \mathbb{Z}) \simeq H_*(S, \mathbb{Z}) \otimes_{\mathbb{Z}} H_*(\Omega_\varrho^I, \mathbb{Z}).$$

Furthermore, we calculate the homology of the domain of discontinuity Ω_ϱ^I:

Theorem B. *Let ϱ and I be as in the previous theorem, and let $\Phi^I \subset G/P$ be the associated model thickening. Then for any integer $k \geqslant 0$ the homology of the domain of discontinuity $\Omega_\varrho^I \subset G/P$ fits in a split exact sequence*

$$0 \to H^{2n-2-k}(\Phi^I, \mathbb{Z}) \to H_k(\Omega_\varrho^I, \mathbb{Z}) \to H_k(\Phi^I, \mathbb{Z}) \to 0,$$

where $n = \dim_{\mathbb{C}} G/P$ is the complex dimension of the flag variety.

The correspondence between Weyl group elements, Schubert cells, and cohomology classes in G/P makes the calculation of the outer terms in the exact sequence above an entirely combinatorial matter. More precisely, we find:

Theorem C. *The domains $\Omega_\varrho^I \subset G/P$ as above have the following properties:*

(i) *The odd homology groups of Ω_ϱ^I vanish.*

(ii) *The even cohomology groups of Ω_ϱ^I are free abelian.*

(iii) *The rank of $H_{2k}(\Omega_\varrho^I)$ is equal to $r_k + r_{n-1-k}$, where $n = \dim_{\mathbb{C}} G/P$ and where r_j denotes the number of elements of I/W_P of length j with respect to the Chevalley-Bruhat order on W/W_P.*

(iv) *For each $k \geqslant 0$ there is a natural isomorphism $H_k(\Omega_\varrho^I, \mathbb{Z}) \simeq H^{2n-2-k}(\Omega_\varrho^I, \mathbb{Z})$.*

Taken together, these results are consistent with the possibility that $\mathcal{W}_\varrho^I$ is a bundle over the surface S with fiber a compact, oriented manifold of dimension $(2n-2)$ homotopy equivalent to Ω_ϱ^I; if so, property (iv) would follow from Poincaré duality for this fiber manifold. We conjecture a weaker form of this:

Conjecture 1.1. *There exists a compact $(2n-2)$-dimensional Poincaré duality space F_ϱ^I homotopy equivalent to Ω_ϱ^I and a continuous fiber bundle*

$$F_\varrho^I \to \mathcal{W}_\varrho^I \to S.$$

In Section 7.6 we verify this conjecture in the case $G = \mathrm{PSL}_3\mathbb{C}$. We have been informed of work in progress by Alessandrini-Li [AL] and Alessandrini-Maloni-Wienhard [AMW] that provides other examples in which Conjecture 1.1 holds.

These homological results also yield a simple formula for the Euler characteristic of the quotient manifold:

Corollary 1.2. *The Euler characteristic of $\mathcal{W}_\varrho^I$ satisfies*

$$\chi(\mathcal{W}_\varrho^I) = \chi(S)\chi(G/P).$$

Note in particular that the Euler characteristic is independent of the choice of balanced ideal $I \subset W$. It also follows that an affirmative answer to Conjecture 1.1 would necessarily produce a fiber space F^I_ϱ which satisfies $\chi(F^I_\varrho) = \chi(G/P)$.

In Section 6, we turn to the complex geometry of quotients. Here we use a technique similar to that of Seade-Verjovsky [SV03] to show that when the limit sets of the representations we consider are sufficiently "small", the associated quotient manifolds $\mathcal{W}^I_\varrho$ inherit holomorphic characteristics from G/P.

In fact, in this part of the paper it is natural for us to work in the more general setting of a complex *semisimple* Lie group G and $N = G/H$ a compact complex homogeneous space (where $H < G$ is a closed complex Lie subgroup). We say that a complex manifold $\mathcal{W}$ is an *embedded* (G, N)*-manifold with data* (Ω, Γ) if:

- There exists a discrete torsion-free group $\Gamma < G$ and a Γ-invariant domain of proper discontinuity $\Omega \subset N$ upon which Γ acts with compact quotient, and
- There is a biholomorphism $\mathcal{W} \simeq \Gamma \backslash \Omega$.

Note that an embedded (G, N)-manifold is a special case of a locally homogeneous geometric structure modeled on (G, N), and that the manifold $\mathcal{W}^I_\varrho$ associated to a right-W_P-invariant ideal I is an embedded $(G, G/P)$-manifold with data $(\Omega^I_\varrho, \varrho(\pi))$.

Following terminology from the study of convex-cocompact group actions, we call $\Lambda := (G/P) - \Omega$ the *limit set* of $\mathcal{W}$. Let m_k denote the k-dimensional Hausdorff measure associated to any Riemannian metric on G/P. Using Shiffman's extension theorem for holomorphic functions (see [Shi68] and Theorem 6.1 below), we show:

Theorem D. *Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ) and limit set $\Lambda = N - \Omega$. Suppose that N is simply connected and that $m_{2n-2}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} N$. If X is a compact Riemann surface of positive genus, then every holomorphic map $\mathcal{W} \to X$ is constant.*

Using this theorem and a result of Beauville and Siu we answer in the negative the question of whether such a manifold admits a Kähler metric under the condition that the fundamental group of the manifold surjects a surface group.

Theorem E. *Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ) and limit set Λ. Suppose that N is simply connected and that $m_{2n-2}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} N$. If there exists a surjective homomorphism $\pi_1\mathcal{W} \to \pi_1 S$ for some closed, orientable surface S of genus $g > 1$, then the complex manifold $\mathcal{W}$ does not admit a Kähler metric. In particular, it is not a complex projective variety.*

In order to apply Theorems D and E to examples arising from Anosov representations, it is necessary to verify the hypothesis concerning the Hausdorff measure of the limit set. We do this in the technical Section 4, which relies on a combinatorial property of balanced ideals in Weyl groups. Namely, except for some low rank aberrations, every balanced ideal $I \subset W$ contains every element $w \in W$ of length at most 2. (We note that a similar result was proved in [ST15] for a similar purpose, but only for a certain class of Chevalley-Bruhat ideals.) This translates to a lack of high-dimension cells in Φ^I, which allows us to show that $m_{2n-2}(\Lambda^I_\varrho)$ vanishes in the G-quasi-Fuchsian case. We conclude:

Theorem F. *Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation, where G is a complex simple adjoint Lie group that is not isomorphic to $\mathrm{PSL}_2\mathbb{C}$, and let $P < G$ be a parabolic subgroup. Let $I \subset W$ a balanced and right-W_P-invariant ideal in the Weyl group. Then the associated compact quotient manifold $\mathcal{W}^I_\varrho$ has the following properties:*

(i) *There is no nonconstant holomorphic map from $\mathcal{W}^I_\varrho$ to a compact Riemann surface of positive genus. In particular, $\mathcal{W}$ is not a holomorphic fiber bundle over a Riemann surface.*

(ii) *The complex manifold $\mathcal{W}^I_\varrho$ does not admit a Kähler metric, and in particular it is not a complex projective variety.*

While Theorems D–F are essentially negative results—they rule out the use of certain techniques in understanding these manifolds—our methods also lead to positive results concerning the behavior of holomorphic line bundles on embedded $(G, G/P)$-manifolds $\mathcal{W} \simeq \Gamma\backslash\Omega$. Specifically, we find that the behavior of such holomorphic line bundles is closely related to the representation theory of the discrete group $\Gamma < G$.

Since every holomorphic line bundle on G/P is G-equivariant, there is a homomorphism

$$p^\Gamma_* : \mathrm{Pic}(G/P) \to \mathrm{Pic}(\mathcal{W})$$

called the *invariant direct image*. In favorable circumstances, the extension theorems of Harvey (see [Har74] and Theorem 6.2 below) allow us to show that $\mathrm{Pic}(\mathcal{W})$ splits as a direct sum, with p^Γ_* being the inclusion of one factor:

Theorem G. *Let G be a semisimple complex Lie group, $P < G$ a parabolic subgroup, and $\mathcal{W}$ an embedded $(G, G/P)$-manifold with data (Ω, Γ) and limit set Λ. Suppose that $m_{2n-4}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} G/P$. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Pic}(\mathcal{W}) \to \mathrm{Pic}(G/P) \to 0 \tag{1.1}$$

which is canonically split by the invariant direct image homomorphism

$$p^\Gamma_* : \mathrm{Pic}(G/P) \to \mathrm{Pic}(\mathcal{W}). \tag{1.2}$$

As before, after excluding some low dimensional cases, this allows us to compute the Picard group of manifolds arising from G-quasi-Fuchsian representations.

Theorem H. *Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation, where G is a complex simple adjoint Lie group that is not of type A_1, A_2, A_3 or B_2. Let $P < G$ be a parabolic subgroup, $I \subset W$ a balanced and right-W_P-invariant ideal in the Weyl group, and $\mathcal{W}_\varrho^I$ the embedded $(G, G/P)$-manifold associated to these data. Then the Picard group $\mathrm{Pic}(\mathcal{W}_\varrho^I)$ satisfies (1.1) and (1.2) for $\Gamma = \varrho(\pi_1 S)$.*

Having calculated the Picard group, in Section 6.3 we turn to calculations of sheaf cohomology groups of line bundles on $\mathcal{W}$ in the image of the invariant direct image homomorphism. Here we restrict to the case $P = B$ to simplify the discussion, though analogous statements could be derived for any parabolic subgroup.

Recall that a holomorphic line bundle $\mathcal{L}$ on G/B is effective if it admits a nontrivial holomorphic section. There exist many effective line bundles on G/B—such bundles are in bijection with the set of dominant weights in the dual of a Cartan subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. We show:

Theorem I. *Let $\mathcal{L}$ be an effective line bundle on G/B and let $\mathcal{W}$ be an embedded $(G, G/B)$-manifold with data (Ω, Γ) and limit set Λ satisfying $m_{2n-2k-2}(\Lambda) = 0$ for some $k \geqslant 1$, where $n = \dim_{\mathbb{C}} G/B$. Then for all $0 \leqslant i < k$,*

$$H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L})) \simeq H^i(\Gamma, H^0(G/B, \mathcal{L})).$$

In this theorem, the expression $H^i(\Gamma, H^0(G/B, \mathcal{L}))$ denotes the group cohomology of Γ. Since $\mathcal{L}$ is G-equivariant and $\Gamma < G$, the space $H^0(G/B, \mathcal{L})$ has the structure of a Γ-module.

When i exceeds the cohomological dimension $\mathrm{cd}(\Gamma)$, the previous theorem becomes the vanishing result:

$$H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L})) = 0 \text{ for } \mathrm{cd}(\Gamma) < i < k. \tag{1.3}$$

We close the discussion of the complex geometry of quotients with the following theorem regarding the existence of meromorphic functions on embedded $(G, G/B)$-manifolds arising from G-quasi-Fuchsian representations. Recall that an ample line bundle $\mathcal{L}$ on G/B is one which gives rise to a projective embedding.

Theorem J. *Suppose G is an adjoint complex simple Lie group not of type A_1, A_2, A_3 or B_2. Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation with image Γ, and let I be a balanced ideal in the Weyl group W of G. Let $\mathcal{W}_\varrho^I$ denote the embedded $(G, G/B)$-manifold associated to these data. For any ample line bundle $\mathcal{L}$ on G/B, the following properties hold:*

(i) *There exists a $k > 0$ such that*

$$H^0(\mathcal{W}_\varrho^I, p_*^\Gamma(\mathcal{L}^k)) \simeq H^0(\Gamma, H^0(G/B, \mathcal{L}^k)) \neq 0.$$

(ii) *The manifold $\mathcal{W}_\varrho^I$ admits a non-constant meromorphic function.*

The same techniques show that the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions on $\mathcal{W}_\varrho^I$ is large whenever the rank of $H^0(\mathcal{W}_\varrho^I, p_*^\Gamma(\mathcal{L}^k))$ is large; however, whether or not there are any cases where this transcendence degree is equal to the complex dimension of $\mathcal{W}_\varrho^I$, so that $\mathcal{W}_\varrho^I$ is Moishezon, is yet to be seen.

1.1 Outline

In Section 2 we recall some facts from Lie theory and introduce the notion of an Anosov representation of a word hyperbolic group.

In Section 3 we review the geometry of flag varieties and discuss the construction of Kapovich-Leeb-Porti which produces domains of proper discontinuity for Anosov representations. For the benefit of readers familiar with [KLP13], we note that in some cases our notation and terminology are different from that of the above-cited paper; this is done to adapt their theory to suit the specific cases we study (for example, we only consider representations into complex Lie groups).

In Section 4 we derive estimates for the Hausdorff dimension of the complement of a domain of discontinuity for an Anosov representation. While these estimates are essential in Section 6, their derivation represents a technical excursion into combinatorial and geometric considerations that are not used elsewhere in the paper. (A reader might skip this section on first reading if seeking an efficient route to the results of Section 6.)

Section 5 contains the main results concerning the topology of domains of discontinuity and of quotient manifolds, including the proofs of Theorems A, B, and C. The results on homology and cohomology of flag varieties from Section 3.4 are used extensively here.

In Section 6 we turn to the complex geometry of quotients, proving Theorems D, E, G, and I on embedded $(G, G/P)$-manifolds, and their consequences for G-quasi-Fuchsian representations, Theorems F, H, and J. The Borel-Bott-Weil theorem and related notions that are used in our analysis of holomorphic line bundles and sheaves on embedded $(G, G/P)$-manifolds are also recalled here. This section does not use the results of Section 5, and could be read independently of that one.

Finally, in Section 7 we present some explicit examples of ideals in the Weyl group. We apply the results of Section 5 to these examples, in some cases obtaining explicit formulas for the Betti numbers of these domains and their quotient manifolds. We also give an alternative description of the unique cocompact domain of discontinuity in G/B for a G-Fuchsian representation $\pi_1 S \to G$ in the case $G = \mathrm{PSL}_3\mathbb{C}$, showing that it is a compactification of a finite quotient of the frame bundle of $S \times \mathbb{R}$. Using this description, we verify that Conjecture 1.1 holds in this case.

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2 Lie groups and Anosov representations

2.1 Real semisimple Lie groups

We use the term *real semisimple Lie group* to mean a Lie group G with finite center, finitely many connected components, no compact factors, and semisimple Lie algebra.

Let G be a real semisimple Lie group. Up to conjugation, there is a unique maximal compact subgroup $K < G$. If $\mathfrak{k}$ is the Lie algebra of K, then the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ induces an $\mathrm{ad}(\mathfrak{k})$-invariant *Cartan decomposition* $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{q}$. A *Cartan subspace* $\mathfrak{a} \subset \mathfrak{q}$ is a maximal abelian subspace.

An element $\alpha \in \mathfrak{a}^* \backslash \{0\}$ is a *restricted root* if the subspace

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \,:\, \mathrm{ad}(Y)(X) = \alpha(Y)(X) \text{ for all } Y \in \mathfrak{a}\}$$

is nonempty; then $\mathfrak{g}_\alpha$ is the associated *root subspace*. Let $\Sigma \subset \mathfrak{a}^*$ denote the set of all restricted roots. The Lie algebra $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} \simeq \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

Choose a partition

$$\Sigma = \Sigma^+ \sqcup \Sigma^-$$

where for each $\alpha \in \Sigma$, exactly one of $\alpha, -\alpha$ lies in Σ^+ and so that if $\alpha, \beta \in \Sigma^+$ and $\alpha + \beta \in \Sigma$, then $\alpha + \beta \in \Sigma^+$. In such a decomposition, we say Σ^+ is the set of *positive* restricted roots and Σ^- is the set of *negative* restricted roots. The *simple* restricted roots $\Delta \subset \Sigma^+$ are those which cannot be expressed as a sum of two elements of Σ^+.

The cardinality ℓ of Δ is independent of the choices made in this construction and is equal to the dimension of $\mathfrak{a}$. This integer is the *real rank* of $\mathfrak{g}$.

A standard *minimal parabolic* subalgebra $\mathfrak{b}$ is defined by

$$\mathfrak{b} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where the nilpotent subalgebra $\mathfrak{n}$ is given by the span of the positive restricted root spaces, $\mathfrak{n} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. We also define $\mathfrak{b}^- = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^-$ where $\mathfrak{n}^- = \bigoplus_{\alpha\in\Sigma^-}$.

Next, let $\Theta \subset \Delta$ be a subset of the simple restricted roots. Let $\Sigma_\Theta^- \subset \Sigma^-$ denote the set of negative roots that can be expressed as an integer linear combination of elements of $\Delta - \Theta$ with nonpositive coefficients. The *standard parabolic subalgebra* $\mathfrak{p}_\Theta \subset \mathfrak{g}$ associated to Θ is defined by:

$$\mathfrak{p}_\Theta := \left(\bigoplus_{\alpha\in\Sigma_\Theta^-} \mathfrak{g}_\alpha \right) \oplus \mathfrak{b}.$$

Similarly, we define $\mathfrak{p}_\Theta^- = \left(\bigoplus_{-\alpha\in\Sigma_\Theta^-} \mathfrak{g}_\alpha\right) \oplus \mathfrak{b}^-$. Thus for example $\mathfrak{p}_\Delta = \mathfrak{b}$ and $\mathfrak{p}_\Delta^- = \mathfrak{b}^-$. The associated *standard parabolic subgroup* is defined by $P_\Theta = N_G(\mathfrak{p}_\Theta)$. The *standard opposite parabolic subgroup* is $P_\Theta^- = N_G(\mathfrak{p}_\Theta^-)$.

A subgroup $P < G$ is *parabolic* if it is conjugate to P_Θ for some $\Theta \subset \Delta$. A pair of parabolic subgroups (P^+, P^-) are *opposite* if their intersection $L = P^+ \cap P^-$ is a maximal reductive subgroup of P^+ and of P^-; in this case we say L is a *Levi factor*. For any $\Theta \subset \Delta$, the pair (P_Θ, P_Θ^-) is opposite. Moreover, given *any* pair of opposite parabolic subgroups (P^+, P^-), there exists a unique Cartan decomposition of $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{q}$, Cartan subspace $\mathfrak{a} \subset \mathfrak{q}$, choice of positive and simple restricted roots Δ, and subset $\Theta \subset \Delta$ such that

$$(P^+, P^-) = (P_\Theta, P_\Theta^-).$$

Relative to the Cartan decomposition, the Weyl group of G is given by

$$W = N_K(\mathfrak{a})/Z_K(\mathfrak{a}).$$

The Weyl group W acts on $\mathfrak{a}$ by the adjoint representation; it has the structure of a finite Coxeter group with respect to the generating set given by reflections in the simple restricted root hyperplanes $V_\alpha := \ker(\alpha) \subset \mathfrak{a}$ for $\alpha \in \Delta$. The induced action of W on $\mathfrak{a}^*$ preserves the set of simple restricted roots Δ and the set of all restricted roots.

There is a unique element $w_0 \in W$, the *longest element*, such that

$$w_0(\alpha) \in -\Delta$$

for every $\alpha \in \Delta$; furthermore, w_0 has order 2. We also define the *opposite involution* $\iota : \Delta \to \Delta$ by $\iota(\alpha) = -w_0(\alpha)$.

Given a subset of simple restricted roots $\Theta \subset \Delta$, the standard parabolic subgroup $P_{\iota(\Theta)}$ is conjugate to P_Θ^-; as a conjugating element one can take any representative $\widetilde{w}_0 \in G$ of the longest element of the Weyl group. Therefore, a standard parabolic subgroup defined by Θ is conjugate to an opposite if and only if

$$\iota(\Theta) = \Theta.$$

We call a parabolic subgroup $P < G$ *symmetric* if it is conjugate to any, hence all, of its opposite parabolic subgroups. We remark that if all simple factors of G are of type A_1, $B_{n\geqslant 2}$, $D_{2k\geqslant 4}$, E_7, E_8, F_4 or G_2, then $\iota = e$ and all parabolic subgroups are symmetric.

If (P^+, P^-) is a pair of opposite parabolic subgroups with common Levi factor $L = P^+ \cap P^-$, then the diagonal G-action on the product $G/P^+ \times G/P^-$ has a unique open orbit, which is G-equivariantly identified with the set of pairs of opposite parabolic subgroups in the conjugacy class of the pair (P^+, P^-). This orbit is also equivariantly isomorphic the homogeneous space G/L.

2.2 Cartan projection

If G is a real semisimple Lie group, the choice of maximal compact subgroup $K < G$ induces a Cartan decomposition $G = KAK$ where $A = \exp(\overline{\mathfrak{a}}^+)$ and

$$\overline{\mathfrak{a}}^+ := \{Y \in \mathfrak{a} \,:\, \alpha(Y) \geqslant 0 \text{ for all } \alpha \in \Sigma^+\}$$

is a closed positive Weyl chamber. Writing an element $g \in G$ in terms of this decomposition

$$g = k \cdot \exp(\mu(g)) \cdot k',$$

the element $\mu(g) \in \overline{\mathfrak{a}}^+$ is uniquely determined. This gives rise to a continuous, proper map

$$\mu : G \to \overline{\mathfrak{a}}^+$$

called the *Cartan projection*.

2.3 Complex semisimple groups

We use the term *complex semisimple Lie group* to mean a complex Lie G group with finitely many connected components and semisimple Lie algebra. Note that such a group is also a real semisimple Lie group by our conventions—a complex semisimple group automatically has finite center[2]. If G is connected and its Lie algebra is simple, we say G is a *complex simple Lie group*.

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. A *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a maximal abelian subalgebra such that the linear map $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is diagonalizable for every $x \in \mathfrak{h}$. Note that this is *not* the same as a Cartan subspace when viewing $\mathfrak{g}$ as a real semisimple Lie algebra. The relationship is as follows: there exists a Cartan decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ for a compact form $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{a} := \mathfrak{h} \cap i\mathfrak{k} \subset i\mathfrak{k}$ constitutes a Cartan subspace. In the notation of the previous section,

$$\mathfrak{h} = \mathfrak{m} \oplus \mathfrak{a} \subset \mathfrak{k} \oplus \mathrm{i}\mathfrak{k}.$$

[2]The center of a complex semisimple Lie group G is discrete and contained in any maximal compact subgroup $K < G$, and thus is finite.

There is a unique Cartan subalgebra up to adjoint action of G. The *rank* of G is the dimension (over $\mathbb{C}$) of any Cartan subalgebra.

Given $\alpha \in \mathfrak{h}^* \backslash \{0\}$, define

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \,:\, \mathrm{ad}(Y)(X) = \alpha(Y)X \text{ for all } Y \in \mathfrak{h}\}.$$

An element $\alpha \in \mathfrak{h}^*$ is a *root* if $\mathfrak{g}_\alpha \neq \{0\}$ and $\mathfrak{g}_\alpha$ is the associated *root subspace*. The set of all roots is denoted by Σ.

A partition $\Sigma = \Sigma^+ \sqcup \Sigma^-$ into positive and negative roots and the associated system $\Delta \subset \Sigma^+$ of simple roots are then defined by the same conditions as in the restricted root setting considered previously. These data define a *Borel subalgebra*

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha,$$

which may also be defined as a maximal solvable subalgebra.

For a subset $\Theta \subset \Delta$, we can as before construct a set $\Sigma^-_\Theta \subset \Sigma^-$ and the associated (complex) standard parabolic subalgebra $\mathfrak{p}_\Theta = \left(\bigoplus_{\alpha \in \Sigma^-_\Theta} \mathfrak{g}_\alpha\right) \oplus \mathfrak{b}$.

We define the corresponding Lie subgroups by

$$\begin{aligned} H &= C_G(\mathfrak{h}), \\ B &= N_G(\mathfrak{b}), \\ P_\Theta &= N_G(\mathfrak{p}_\Theta). \end{aligned}$$

The subgroup $H < G$ is called a *Cartan subgroup*, and is a maximal torus[3] in G. General parabolic subgroups are then defined analogously to the real case, as objects conjugate to the respective standard examples. As in the real case, two parabolic subgroups P^+, P^- are *opposite* if $P^+ \cap P^- = L$ is a Levi subgroup of both P^+, P^-.

If $\mathfrak{g}$ is a complex semisimple Lie algebra, then a *split real form* $\mathfrak{g}_\mathbb{R}$ is a real form of the Lie algebra such that any Cartan subspace $\mathfrak{a} \subset \mathfrak{g}_\mathbb{R}$ complexifies to a Cartan subalgebra of $\mathfrak{g}$, which occurs if and only if the real rank of $\mathfrak{g}_\mathbb{R}$ is equal to the rank of $\mathfrak{g}$. There is a single equivalence class of split real forms under the adjoint G-action on $\mathfrak{g}$; choosing a representative of this class, we refer to *the* split real form $\mathfrak{g}_\mathbb{R} \subset \mathfrak{g}$. The connected Lie subgroup $G_\mathbb{R} < G$ with lie algebra $\mathfrak{g}_\mathbb{R}$ is the split real form of G.

2.4 Principal three-dimensional subgroups

For more information on the objects in this section, see the discussion in [ST15] and the original paper of Kostant [Kos59].

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ. Choose a nilpotent element $e_1 \in \mathfrak{g}$ which has ℓ-dimensional centralizer (a *regular nilpotent*). By the Jacobson-Morozov theorem

[3] A maximal torus $H < G$ is a maximal abelian subgroup which is isomorphic to $(\mathbb{C}^*)^{\mathrm{rank}(G)}$.

([Jac51] [Mor42]), there exists elements $x, f_1 \in \mathfrak{g}$ such that the triple $\{f_1, x, e_1\}$ spans a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$, with f_1, x, and e_1 respectively corresponding to $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Such a subalgebra $\mathfrak{s}$ is called a *principal three-dimensional subalgebra*. There is a single conjugacy class of principal three-dimensional subalgebras under the adjoint G-action on $\mathfrak{g}$, corresponding to the single conjugacy class of regular nilpotents. Abusing terminology, we use this uniqueness to refer to *the* principal three-dimensional subalgebra of $\mathfrak{g}$.

If $G \simeq \mathrm{Aut}_0(\mathfrak{g})$ is the adjoint complex simple group associated to $\mathfrak{g}$, and $\mathfrak{s} \subset \mathfrak{g}$ is the principal three-dimensional subalgebra, then associated to the isomorphism $\mathfrak{sl}_2\mathbb{C} \simeq \mathfrak{s}$ described above is a unique injective homomorphism

$$\iota_G : \mathrm{PSL}_2\mathbb{C} \to G$$

Moreover, the restriction of ι_G to $\mathrm{PSL}_2(\mathbb{R})$ takes values in the split real form of G. The image $\mathfrak{S}$ of this homomorphism is the *principal three-dimensional subgroup* of G.

Given a maximal torus and Borel subgroup $H_{\mathfrak{S}} < B_{\mathfrak{S}} < \mathfrak{S}$ in the principal three-dimensional subgroup, there exists a unique maximal torus and Borel subgroup $H < B < G$ in G such that $H_{\mathfrak{S}} < H$ and $B_{\mathfrak{S}} < B$. When considering the principal three-dimensional subgroup, we always assume that the maximal tori and Borel subgroups for $\mathfrak{S}$ and G have been chosen in this compatible way. We further assume that the isomorphism ι_G is chosen so that $H_{\mathfrak{S}}$ and $B_{\mathfrak{S}}$ correspond, respectively, to the set of diagonal and upper-triangular matrices in $\mathrm{PSL}_2\mathbb{C}$. Then, identifying the quotient of $\mathrm{PSL}_2\mathbb{C}$ by its upper-triangular subgroup with $\mathbb{P}^1_{\mathbb{C}}$ we obtain an equivariant holomorphic embedding

$$f_G : \mathbb{P}^1_{\mathbb{C}} \simeq \mathfrak{S}/B_{\mathfrak{S}} \to G/B$$

called the *principal rational curve* following [ST15]. The principal rational curve can also be characterized as the unique closed orbit of the action of $\mathfrak{S}$ on G/B.

Since B is self normalizing, the space G/B is equivariantly isomorphic to the space of Borel subgroups of G where G acts on the latter space by conjugation. Using this isomorphism, two points $p, p' \in G/B$ are defined to be *opposite* if the corresponding Borel subgroups are opposite. More generally, a pair of points points $p \in G/P^+$ and $p' \in G/P^-$ corresponds to a pair of parabolic subgroups conjugate, respectively, to P^+ and P^-; we say in this case that p, p' are opposite if the corresponding parabolic subgroups are opposite.

We will need the following essential property of the principal rational curve.

Proposition 2.1. *Given distinct points $z, z' \in \mathbb{P}^1_{\mathbb{C}}$, the images $f_G(z), f_G(z') \in G/B$ are opposite.*

Proof. The statement is invariant under conjugation of $\mathfrak{S}$ by elements of G, hence we can fix a convenient choice of principal three-dimensional subalgebra $\mathfrak{s} = \mathrm{span}(e_0, x_0, f_0)$ as in [ST15, Proposition 1.1] so that $\alpha(x_0) = 2$ for all $\alpha \in \Delta$. Recall that in terms of the derivative of ι_G, the element x_0 is given by $(\iota_G)_* \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let $H_0 \subset \mathrm{PSL}_2\mathbb{C}$ denote the diagonal subgroup. Identify the Weyl group of $\mathrm{PSL}_2\mathbb{C}$ with $\mathbb{Z}/2$, with the nontrivial element represented by $u = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Since u normalizes H_0, the image $\iota_G(u)$ normalizes $\iota_G(H_0)$. Since H is the unique maximal torus containing $\iota_G(H_0)$, it follows that $\iota_G(u) \in N_G(H)$. Thus ι_G induces a homomorphism $W(\mathrm{PSL}_2\mathbb{C}) = N_{\mathrm{PSL}_2\mathbb{C}}(H_0)/H_0 \to W(G) = N_G(H)/H$.

Next we claim that the image of u under this map is the longest element $w_0 \in W = W(G)$. This element is uniquely characterized by the condition that it maps every simple root to a negative root. Note that $\mathrm{Ad}(\iota_G(u))(x_0) = -x_0$. Thus for each $\alpha \in \Delta$ we have

$$\iota_G(u)(\alpha)(x_0) = \alpha(\mathrm{Ad}(\iota_G(u))(x_0)) = \alpha(-x_0) = -2$$

It follows that when expressing $\iota_G(u)(\alpha)$ as a linear combination of the simple roots, there is exactly one nonzero coefficient, which is equal to -1. Hence $\iota_G(u)(\alpha)$ is a negative simple root for all $\alpha \in \Delta$, and we conclude $\iota_G(u)$ represents w_0.

Since the longest element of W maps B to an opposite Borel, it follows from ι_G-equivariance of the map f_G that if $z_0 \in \mathbb{P}^1_{\mathbb{C}}$ is the unique point such that $f_G(z_0) = eB \in G/B$, then $f_G(z_0)$ and $f_G(uz_0) = \iota_G(u) f_G(z_0)$ are opposite. Finally, since $\mathrm{PSL}_2\mathbb{C}$ acts transitively on pairs of distinct points in $\mathbb{P}^1_{\mathbb{C}}$, equivariance of f_G implies that the same condition holds for all pairs of distinct points. $\square$

We also have the important extension of this to other parabolic subgroups.

Corollary 2.2. *Let $B^+_{\mathfrak{S}}, B^-_{\mathfrak{S}}$ be a pair of opposite Borel subgroups of the principal three-dimensional subgroup $\mathfrak{S} < G$, and let B^+, B^- be the respective Borel subgroups of G containing them. Denote by*

$$f^{\pm}_G : \mathrm{PSL}_2\mathbb{C}/B^{\pm}_{\mathfrak{S}} \to G/B^{\pm}$$

the corresponding principal rational curves. Let $P^{\pm}$ be a pair of opposite parabolic subgroups such that $B^+ < P^+$ and $B^- < P^-$. Consider the induced G-equivariant projections

$$p_{\pm} : G/B^{\pm} \to G/P^{\pm}.$$

Then if $z \neq z' \in \mathbb{P}^1_{\mathbb{C}}$, the parabolic subgroups representing the points $p_+ \circ f^+_G(z) \in G/P^+$ and $p_- \circ f^-_G(z') \in G/P^-$ are opposite.

Proof. The combined projection $p_+ \times p_- : G/B^+ \times G/B^- \to G/P^+ \times G/P^-$ is G-equivariant and open. Therefore, the unique open G-orbit is mapped to the unique open G-orbit. Since this open orbit identifies with the set of opposite parabolic subgroups, the result follows from Proposition 2.1. $\square$

2.5 Anosov representations

In this subsection we recall the definition of an Anosov representation and some related notions that are used extensively in the sequel. We follow the exposition of [GGKW] quite closely. Additional background on Anosov representations can be found in [Lab06], [GW12], [KLP13], and [KLP14].

Let d_π denote the word metric on the Cayley graph of π corresponding to some finite generating set. Recall that π is *word hyperbolic* if this Cayley graph is a Gromov hyperbolic metric space. Write $|\cdot|$ for the associated word length function, i.e. $|\gamma| = d_\pi(e, \gamma)$. The *translation length* of $\gamma \in \pi$ is defined by

$$\ell_\pi(\gamma) := \inf_{\beta \in \pi} |\beta\gamma\beta^{-1}|.$$

We denote by $\partial_\infty \pi$ the Gromov boundary of the Cayley graph of π; points in $\partial_\infty \pi$ are equivalence classes of geodesic rays in the Cayley graph. The π-action by left translation on its Cayley graph extends to a continuous action on $\partial_\infty \pi$. Under this action, each infinite order element $\gamma \in \pi$ has a unique attracting fixed point $\gamma^+ \in \partial_\infty \pi$ and a unique repelling fixed point $\gamma^- \in \partial_\infty \pi$.

Let (P^+, P^-) be a pair of opposite parabolic subgroups of G. Let $\varrho : \pi \to G$ be a homomorphism and suppose there exists a pair of continuous, ϱ-equivariant maps

$$\xi^\pm : \partial_\infty \pi \to G/P^\pm.$$

The pair (ξ^+, ξ^-) is *dynamics preserving* for ϱ if for each infinite order element $\gamma \in \pi$ the point $\xi^+(\gamma^+)$ (resp. $\xi^-(\gamma^+)$) is an attracting fixed point for the action of $\varrho(\gamma)$ on G/P^+ (resp. G/P^-). Here, a fixed point $x \in G/P$ is attracting for $g \in G$ if the linear map given by the differential

$$dg_x : T_x G/P \to T_x G/P$$

has spectral radius strictly less than one.

We now come to the definition of an *Anosov* representation.

Definition 2.3. *Let (P^+, P^-) be a pair of opposite parabolic subgroups of G, and $\varrho : \pi \to G$ a homomorphism. Then ϱ is* (P^+, P^-)-Anosov *if there exists a pair of ϱ-equivariant, continuous maps*

$$\xi^\pm : \partial_\infty \pi \to G/P^\pm$$

such that the following conditions hold:

(i) *For all distinct pairs $t, t' \in \partial_\infty \pi$, the points $\xi^+(t) \in G/P^+$ and $\xi^-(t') \in G/P^-$ are opposite.*

(ii) *The pair of maps (ξ^+, ξ^-) is dynamics-preserving for ϱ.*

(iii) *Realize* (P^+, P^-) *as a pair of standard opposite parabolics* (P_Θ, P_Θ^-) *for suitable choices of Cartan subspace* $\mathfrak{a}$*, system of positive roots* Σ^+*, and subset* $\Theta \subset \Delta$*. Then for each* $\alpha \in \Theta$*, any sequence* $\{\gamma_n\} \subset \pi$ *with divergent word length*

$$\limsup_{n\to\infty} \ell_\pi(\gamma_n) \to \infty,$$

satisfies the following α-divergence condition *of the Cartan projections of its* ϱ*-images:*

$$\limsup_{n\to\infty} \langle \alpha, \mu(\varrho(\gamma_n))\rangle = \infty.$$

In the last condition $\langle \cdot, \cdot \rangle$ *denotes the evaluation pairing* $\mathfrak{a}^* \times \mathfrak{a} \to \mathbb{R}$*.*

Due to the work of [GW12], [GGKW] and [KLP13], [KLP14], there are now many equivalent definitions of Anosov representations; we have stated the most economical one for our purposes. If G is a complex Lie group, we understand the Anosov property to be defined with respect to the underlying real Lie group.

The maps $\xi^\pm : \partial_\infty \pi \to G/P^\pm$ in the definition above are called the *limit curves* of the Anosov representation.

If P is a symmetric parabolic subgroup, we can apply the definition above with $(P^+, P^-) = (P, gPg^{-1})$ as the pair of opposite parabolic subgroups. In this case both spaces $G/P^\pm$ are canonically and G-equivariantly identified with G/P, and the limit maps $\xi^\pm$ are related to one another by this identification. We therefore consider such a representation to have a single limit curve

$$\xi : \partial_\infty \pi \to G/P,$$

and in this situation we simply say that ϱ is P*-Anosov.*

The following property of Anosov representations follows quickly from the definitions.

Proposition 2.4. *Let* $P, Q < G$ *be symmetric parabolic subgroups such that* $P < Q$*. If* $\varrho : \pi \to G$ *is* P*-Anosov, then* ϱ *is also* Q*-Anosov. Furthermore, if* $\xi : \partial_\infty \pi \to G/P$ *is the limit curve for* ϱ *as a* P*-Anosov representation, then* $p \circ \xi : \partial_\infty \pi \to G/Q$ *is the limit curve for* ϱ *as a* Q*-Anosov representation where* $p : G/P \to G/Q$ *is the natural projection.*

There is also no loss of generality in considering only P-Anosov representations for symmetric parabolics P rather than the *a priori* more general classes of (P^+, P^-)-Anosov representations:

Proposition 2.5 ([GW12])**.** *Let* $\varrho : \pi \to G$ *be* (P^+, P^-)*-Anosov. Then there exists a symmetric parabolic subgroup* $P < G$ *such that* ϱ *is* P*-Anosov.* □

Furthermore, the following theorem of Guichard-Wienhard [GW12] establishes some basic properties of Anosov representations:

Theorem 2.6. *Let $\varrho : \pi \to G$ be (P^+, P^-)-Anosov. Then the following properties are satisfied:*

(i) *For every $\gamma \in \pi$, the holonomy $\varrho(\gamma)$ is conjugate to an element of $L = P^+ \cap P^-$.*

(ii) *The representation ϱ is discrete, has finite kernel, and is a quasi-isometric embedding.*

(iii) *The set $\mathcal{A}$ of all (P^+, P^-)-Anosov representations of π is an open set in the representation variety $\mathrm{Hom}(\pi, G)$.*

(iv) *The map taking a (P^+, P^-)-Anosov representation to either of its limit curves,*

$$\begin{aligned} \mathcal{A} &\to C^0(\partial_\infty \pi, G/P^\pm) \\ \varrho &\mapsto \xi_\varrho^\pm \end{aligned}$$

is continuous, where $C^0(\partial_\infty \pi, G/P^\pm)$ has the uniform topology. □

In the case that G has real rank one, it was also shown in [GW12] that the Anosov property reduces to the well-known class of *convex-cocompact* subgroups of G:

Theorem 2.7 ([GW12])**.** *Suppose G has real rank one. Then a representation $\varrho : \pi \to G$ is Anosov if and only if ϱ has finite kernel and its image is convex-cocompact.* □

In particular, if Γ is a uniform lattice in a real rank one Lie group G (e.g. $\mathrm{SO}(n,1), \mathrm{SU}(n,1)$), then the inclusion $\Gamma \hookrightarrow G$ is an Anosov representation.

2.6 Fuchsian and Hitchin representations

Let S be a closed, oriented surface of genus at least two. For a Lie group G we define the character variety of S in G to be the set

$$\chi(S, G) = \mathrm{Hom}(\pi_1 S, G)/G$$

where G acts on $\mathrm{Hom}(\pi_1 S, G)$ by conjugation.

Identify the hyperbolic plane $\mathbb{H}^2$ with the upper half plane $\mathbb{H} \subset \mathbb{C}$ (which is oriented by its complex structure). Then $\mathrm{PSL}_2\mathbb{R}$ is identified with the group of orientation preserving isometries of $\mathbb{H}^2$. A *Fuchsian representation* is an injective homomorphism

$$\eta : \pi_1 S \to \mathrm{PSL}_2(\mathbb{R})$$

with discrete image such that the associated homotopy equivalence $S \simeq \varrho(\pi_1 S)\backslash\mathbb{H}^2$ is orientation-preserving.

Let G be a complex simple Lie group of adjoint type and fix a principal three-dimensional subgroup (with embedding $\iota_G : \mathrm{PSL}_2\mathbb{C} \to G$). Let $G_\mathbb{R} < G$ be a split real form which contains $\iota_G(\mathrm{PSL}_2\mathbb{R})$. A representation $\varrho : \pi_1 S \to G$ is $G_\mathbb{R}$-*Fuchsian* if there exists a Fuchsian representation η such that ϱ is conjugate to $\iota_G \circ \eta$. The set conjugacy classes of $G_\mathbb{R}$-Fuchsian representations forms a connected subset of $\chi(S, G_\mathbb{R})$ that is in natural bijection with the Teichmüller space of hyperbolic structures on S.

A $G_\mathbb{R}$-*Hitchin* representation is a homomorphism $\varrho : \pi_1 S \to G_\mathbb{R}$ whose conjugacy class lies in the same path component of $\chi(S, G_\mathbb{R})$ as the $G_\mathbb{R}$-Fuchsian representations. Let $\mathcal{H}(S, G_\mathbb{R}) \subset \chi(S, G_\mathbb{R})$ denote the set of conjugacy classes of $G_\mathbb{R}$-Hitchin representations.

The following theorem organizes the key properties of Hitchin representations which will use.

Theorem 2.8. *When considered as a subset of the $G_\mathbb{R}$-character variety, the set of $G_\mathbb{R}$-Hitchin representations*

$$\mathcal{H}(S, G_\mathbb{R}) \subset \chi(S, G_\mathbb{R})$$

is a smooth manifold diffeomorphic to a Euclidean space of real dimension $-\chi(S) \dim_\mathbb{R}(G_\mathbb{R})$ and is a connected component of $\chi(S, G_\mathbb{R})$. Moreover, every $G_\mathbb{R}$-Hitchin representation is Anosov with respect to a minimal parabolic subgroup $B < G_\mathbb{R}$.

Furthermore, when considered as a representation in the complex group G, each Hitchin representation is a smooth point of the complex affine variety $\mathrm{Hom}(\pi_1 S, G)$.

Proof. The statement that $\mathcal{H}(S, G_\mathbb{R}) \subset \chi(S, G_\mathbb{R})$ is a smooth manifold of the given dimension was proved by Hitchin in [Hit92]. When $G_\mathbb{R} = \mathrm{PSL}_n\mathbb{R}$, Labourie [Lab06] established that Hitchin representations are B-Anosov. For general split groups, the analogous statement was proved by Fock-Goncharov in [FG06, Theorem 1.15]; also see [GW12, Theorem 6.2] for further discussion.

By [Gol84, pg. 204], a representation $\varrho \in \mathrm{Hom}(\pi_1 S, G)$ lies in the smooth locus if and only if it has discrete centralizer. Hitchin representations are irreducible (i.e. not conjugate into a proper parabolic subgroup of G, see [Lab06, Lemma 10.1]), which implies that their centralizers are finite extensions of the center of G ([Sik12, Proposition 15]) and thus discrete. □

2.7 Quasi-Fuchsian and quasi-Hitchin representations

As before, let S be a closed, oriented surface of genus at least two. A representation $\eta : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ is *quasi-Fuchsian* if it is obtained from a Fuchsian representation by a quasiconformal deformation. This is equivalent to being a convex-cocompact representation, or to the existence of a continuous, equivariant, injective map $\xi_\eta : \partial_\infty \pi_1 S \to \mathbb{P}^1_\mathbb{C}$. A quasi-Fuchsian representation is Fuchsian if and only if it is conjugate to a representation with values in $\mathrm{PSL}_2\mathbb{R} < \mathrm{PSL}_2\mathbb{C}$. The space of all quasi-Fuchsian representations up to conjugacy will be denoted

$$\mathcal{QF}(S) \subset \chi(S, \mathrm{PSL}_2\mathbb{C})$$

and the set of Fuchsian representations by

$$\mathcal{F}(S) \subset \chi(S, \mathrm{PSL}_2\mathbb{R}).$$

Now, let G be a complex simple Lie group of adjoint type. A *G-quasi-Fuchsian representation* $\varrho : \pi_1 S \to G$ is a representation which admits a factorization $\varrho = \iota_G \circ \eta$ where η is a quasi-Fuchsian representation. Similarly, a subgroup $\Gamma < G$ is G-quasi-Fuchsian if it is the image of a G-quasi-Fuchsian representation.

The chosen principal three-dimensional embedding $\iota_G : \mathrm{PSL}_2\mathbb{C} \to G$ induces a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\iota_G \circ} & \chi(\pi_1 S, G_{\mathbb{R}}) \\ \downarrow & & \downarrow \\ \mathcal{QF}(S) & \xrightarrow{\iota_G \circ} & \chi(\pi_1 S, G). \end{array} \tag{2.1}$$

Moreover, these maps are independent of the choice of three-dimensional subalgebra and split real form.

We now show that a G-quasi-Fuchsian representation is Anosov and identify the limit curve.

Proposition 2.9. *Every G-quasi-Fuchsian representation ϱ is P-Anosov where $P < G$ is any symmetric parabolic subgroup. Furthermore, if $\varrho = \iota_G \circ \eta$ where $\eta : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ is quasi-Fuchsian, and if η has limit curve $\xi : \partial_\infty \pi_1 S \to \mathbb{P}^1_{\mathbb{C}}$, then the limit curve of ϱ is given by*

$$f_G \circ \xi : \partial_\infty \pi_1 S \to G/P$$

where $f_G : \mathbb{P}^1_{\mathbb{C}} \to G/P$ is the principal rational curve. □

This proposition can be proved using the criterion in [GW12] regarding when an Anosov representation remains Anosov after composing with a homomorphism to a larger Lie group, but we include a sketch of a proof here to give some indication of how Definition 2.3 is applied.

Proof. Firstly, by Proposition 2.4, if we show that the above statement is true for a Borel subgroup $P = B$, then the result follows for all other symmetric parabolics.

By Proposition 2.1, the composition

$$f_G \circ \xi : \partial_\infty \pi_1 S \to G/B$$

satisfies property (i) of Definition 2.3. For property (ii) of the definition, we use conjugation in G to affect the same normalization of $\mathfrak{S}$ considered in Proposition 2.1, where $\alpha(x_0) = 2$ for all $\alpha \in \Delta$ and $x_0 \in \mathfrak{g}$ is the semisimple element of the $\mathfrak{sl}_2$-triple generating the principal three dimensional subalgebra. For any nontrivial element $\gamma \in \pi_1 S$ we can assume (after conjugating η in $\mathrm{PSL}_2\mathbb{C}$) that $\eta(\gamma) = \exp\left(\zeta \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right) = \begin{pmatrix} e^{\zeta} & 0 \\ 0 & e^{-\zeta} \end{pmatrix}$ for $\zeta \in \mathbb{C}$ with $\mathrm{Re}(\zeta) > 0$. Thus $\xi(\gamma_+) = z_0$ satisfies $f_G(z_0) = eB$ and $\varrho(\gamma) = \exp(\zeta x_0)$. Then

$$T_{eB}(G/B) \simeq \bigoplus_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha$$

and this is a decomposition into eigenspaces for the action of $\varrho(\gamma)$, where the eigenvalue on $\mathfrak{g}_\alpha$ is $\exp(\alpha(\zeta x_0))$. Since $\alpha(x_0) = 2$ for $\alpha \in \Delta$, for $\alpha \in \Sigma^-$ we have $\alpha(x_0) < 0$ and $|\exp(\alpha(\zeta x_0))| < 1$. This verifies that eB is the attracting fixed point for $\varrho(\gamma)$, and property (ii) of Definition 2.3 follows.

Finally, for property (iii) of Definition 2.3, we note that for any divergent sequence of regular semisimple elements $\{g_n\} \subset \mathrm{PSL}_2\mathbb{C}$, their images under the principal three-dimensional embedding ι_G satisfy

$$\lim_{n\to\infty} \langle \mu(\iota_G(g_n)), \alpha\rangle = \infty$$

for every simple root $\alpha \subset \Delta$. Since every element in the image of η is regular semisimple, this verifies property (iii) and completes the proof. □

Using Theorem 2.7 and the equivalence of quasi-Fuchsian and convex-compact for representations $\pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ it follows that:

Corollary 2.10. *The set of B-Anosov representations $\varrho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ is equal to the set of quasi-Fuchsian representations.* □

It is a consequence of the *Bers Simultaneous Uniformization Theorem* [Ber60] that in this case, the space of B-Anosov representations is connected, and moreover biholomorphic to

$$\mathcal{QF}(S) \simeq \mathcal{T}(S) \times \mathcal{T}(\overline{S})$$

where $\mathcal{T}(S)$ is the Teichmüller space of isotopy classes of complex structures on S which agree with the orientation, and $\overline{S}$ denotes S with the opposite orientation.

Let P be a symmetric parabolic subgroup of G. We define the space of (G, P)-quasi-Hitchin representations

$$\mathcal{QH}(S, G, P) \subset \chi(S, G)$$

as the connected component of P-Anosov representations which contains the Hitchin representations. For later use, we denote the preimage of this set under the quotient mapping $\mathrm{Hom}(\pi_1 S, G) \to \chi(S, G)$ by

$$\widetilde{\mathcal{QH}}(S, G, P) \subset \mathrm{Hom}(\pi_1 S, G).$$

By Theorem 2.7, there is an equality

$$\mathcal{QF}(S) = \mathcal{QH}(S, \mathrm{PSL}_2\mathbb{C}, B)$$

since quasi-Fuchsian space is the connected component of B-Anosov representations which contains the space of Fuchsian representations $\mathcal{F}(S) = \mathcal{H}(S, \mathrm{PSL}_2\mathbb{R})$.

3 Flag varieties and the KLP construction

In this section, we will explain in some detail the construction of Kapovich-Leeb-Porti of domains of discontinuity for Anosov representations. Our account differs from that of [KLP13] in that we focus on complex semisimple Lie groups and their associated flag varieties and avoid the discussion of visual boundaries of symmetric spaces. This presentation is tailored to the application of sections 4 and 5.

3.1 Length function and Chevalley-Bruhat order

References for the following standard material include [Bou02] and [BB05].

Let G be a complex semisimple Lie group. As in Section 2 let W denote the Weyl group of G associated to a maximal torus $H < G$. Fix a system Δ of simple roots and let $S = \{r_\alpha : \alpha \in \Delta\}$ denote the associated system of reflection generators for W. Then (W, S) is a Coxeter system and hence gives rise to a partial order $<$ on W, the *Chevalley-Bruhat order*, and a *length function*

$$\ell : W \to \mathbb{Z}^{\geqslant 0}$$

where $\ell(w)$ is the minimum length of w as a word in the generating set S. A representation of w as a word in S of length $\ell(w)$ is said to be *reduced*. The function ℓ is the rank function for the Chevalley-Bruhat order, which means in particular that $x < y$ implies $\ell(x) < \ell(y)$. Inversion in W preserves these structures, i.e. $\ell(w^{-1}) = \ell(w)$ and $x < y$ if and only if $x^{-1} < y^{-1}$. When $a < b$ for $a, b \in W$, we say that b *dominates* a. In the usual way we use $\leqslant$ to denote the associated nonstrict comparison operation of the Chevalley-Bruhat order.

The longest element $w_0 \in W$ was introduced in Section 2 and defined relative to its action on the roots; equivalently, w_0 is the unique element of W on which the function ℓ attains its maximum. Multiplication by w_0 on the left defines an antiautomorphism of the Chevalley-Bruhat order and length function; that is, it inverts length and comparisons:

$$\ell(w_0 w) = \ell(w_0) - \ell(w) \quad \text{and} \quad (a < b) \Leftrightarrow (w_0 b < w_0 a). \tag{3.1}$$

Now let $P < G$ be a standard parabolic subgroup. The *Weyl group of* P is defined as

$$W_P = (N_G(H) \cap P)/H.$$

Note that $W_P < W = N_G(H)/H$, and that if $P = B$ we have $W_P = \{e\}$. The space W/W_P of left W_P-cosets inherits a partial order from that of W as follows: Each coset wW_P has a unique minimal element, and letting W^P denote the set of such minimal elements, we have a canonical bijection $W/W_P \simeq W^P$. Restricting the Chevalley-Bruhat order to W^P gives the desired partial order on W/W_P. Extending the previous terminology, we also call this order on W/W_P the Chevalley-Bruhat order, and we call the resulting rank function the length function on W/W_P. Explicitly, the latter function is

$$\ell : W/W_P \to \mathbb{Z}^{\geqslant 0}, \;\; \ell(wW_P) := \min_{w' \in wW_P} \ell(w').$$

We also note that the length function on W/W_P satisfies

$$\ell(w_0 w W_P) = \ell(w_0 W_P) - \ell(w W_P),$$

and $\ell(w_0 W_P)$ is the maximum value of ℓ on W/W_P.

There is a further extension of the Chevalley-Bruhat order for a pair P, Q of standard parabolic subgroups of G: Each double coset in $W_P \backslash W / W_Q$ contains a unique minimal element, and restricting the Chevalley-Bruhat order to the set $W^{P,Q}$ of such minimal elements gives a partial order on $W_P \backslash W / W_Q$.

3.2 Chevalley-Bruhat ideals

A *Chevalley-Bruhat ideal* (or briefly, an *ideal*) is a subset $I \subset W$ such that if $b \in I$ and $a < b$, then $a \in I$. That is, I is *downward closed* for the partial order. (In [KLP13] ideals are called *thickenings*, though several other objects are given that name as well; we reserve the term thickening for a subset of the flag variety that is defined below.) Associated to any element $x \in W$ there is the *principal ideal* defined as $\langle x \rangle = \{w \in W \,:\, w \leqslant x\}$. It is easy to see that every ideal $I \subset W$ is a union of principal ideals, and in fact has a unique minimal description $I = \bigcup_{i=1}^{r} \langle x_i \rangle$ as a union of principal ideals. The elements x_i appearing in this minimal presentation are exactly those which lie in I but are not dominated by any element of I. We call $\{x_1, \ldots, x_r\}$ the *minimal generating set* of I.

If $I \subset W$ is an ideal, then $I^{-1} = \{x^{-1} \,:\, x \in I\}$ is also an ideal. The complement of a nonempty ideal is never an ideal, however if we define

$$I^{\perp} = w_0(W - I)$$

then, by the antiautomorphism property of $w \mapsto w_0 w$, we find that $I^{\perp}$ is an ideal. We call this the *orthogonal* of I. Note that it is always the case that

$$W = I \sqcup w_0 I^{\perp}.$$

Following the terminology of [KLP13], we say that an ideal $I \subset W$ is *slim* if $I \subset I^{\perp}$, *fat* if $I \supset I^{\perp}$, and *balanced* if $I = I^{\perp}$ (equivalently, if it is both fat and slim). Note in particular that a balanced ideal satisfies $|I| = \frac{1}{2}|W|$, and that for slim ideals, this cardinality condition is equivalent to being balanced.

3.3 Flag variety and Schubert cells

We now discuss the cell structures of flag varieties in relation to the Weyl group and the Chevalley-Bruhat order; this material is standard and can be found in e.g. [BGG73] [Ful97] [LG01] [CG10].

Let $B < G$ be the Borel subgroup associated to the choice Δ of simple roots fixed above. The homogeneous space G/B is the *full flag variety* of G. If $P \subset G$ is a parabolic subgroup, then G/P is the *partial flag variety* associated to P. All flag varieties are smooth projective varieties over $\mathbb{C}$, and in particular are compact oriented manifolds.

The full flag variety G/B has a natural decomposition into a disjoint union of B-orbits called *Schubert cells*

$$\{C_w = BwB \ : \ w \in W\}.$$

Each C_w is diffeomorphic to $\mathbb{C}^{\ell(w)}$. The closure $X_w = \overline{C_w}$ is a *Schubert variety* and can be described as the union of the cells that are dominated by w in the Chevalley-Bruhat order:

$$X_w = \{C_{w'} \ : \ w' \leqslant w\}.$$

Therefore, there is a bijection between W and the set of Schubert cells, where ideals $I \subset W$ correspond to unions of Schubert varieties. In topological terms, ideals $I \subset W$ are in bijection with closed, cellular subcomplexes of G/B with respect to the cellular structure given by the Schubert cells. In algebraic terms, Schubert varieties are irreducible projective subvarieties of G/B.

For a parabolic $P < G$ containing B, we have the projection $\pi : G/B \to G/P$. Under this projection, the Schubert cell decomposition of G/B projects to a cell decomposition of G/P, and the projection of a Schubert cell C_w to G/P depends only on the coset $wW_P \in W/W_P$. Thus, the cells in G/P are indexed by the coset space W/W_P, or by the collection of coset representatives W^P. We define

$$\begin{aligned} C_{wW_P} &:= \pi(C_w) \\ X_{wW_P} &:= \overline{C_{wW_P}}. \end{aligned}$$

The set X_{wW_P} is called a Schubert variety in G/P, and is an irreducible projective subvariety. As before, the Chevalley-Bruhat order (now on W/W_P) is equivalent to the inclusion partial order on these Schubert varieties. Note that the real dimension of G/B is $2\ell(w_0)$, while that of G/P is $2\ell(w_0 W_P)$.

The Schubert cells are defined as B-orbits in flag varieties of G. In what follows, we will also need to understand the structure of the P-orbits on G/Q for P, Q parabolic subgroups. We summarize the results in the following (see [Per02] and [Pec14]):

Theorem 3.1.

(i) *Every P-orbit in G/Q can be written as PwQ for some $w \in W$.*

(ii) *This description gives a bijection between the set of P-orbits in G/Q and the double cosets $W_P \backslash W / W_Q$, where W_P and W_Q denote the Weyl groups of P and Q.*

(iii) *The inclusion partial order on closures of P-orbits in G/Q corresponds, under this bijection, to the Chevalley-Bruhat order on $W_P \backslash W / W_Q$.*

(iv) *Each P-orbit is a union of B-orbits; specifically, we have*

$$PwQ = \bigcup_{(w_P, w_Q) \in W_P \times W_Q} Bw_P w w_Q Q. \tag{3.2}$$

□

3.4 Homology and cohomology of the flag variety

As above let P be a parabolic subgroup of G, a complex semisimple Lie group. The integral homology $H_*(G/P, \mathbb{Z})$ is naturally isomorphic to the free $\mathbb{Z}$-module $\mathbb{Z}[W/W_P]$ with grading given by twice the length function, 2ℓ. This can be seen using cellular homology for the Schubert cell decomposition of G/P; then $\mathbb{Z}[W/W_P]$ with grading 2ℓ is the cellular chain complex, and the boundary maps are zero since all cells have even dimension. Concretely, in this isomorphism the element $wW_P \in W/W_P$ corresponds to the cell C_{wW_p} (in the cellular resolution), or to the fundamental class $[X_{wW_P}] \in H_{2\ell(wW_P)}(G/P, \mathbb{Z})$ of the Schubert variety X_{wW_P}.

Correspondingly, the universal coefficients theorem identifies $H^*(G/P, \mathbb{Z})$ with the dual space $\mathbb{Z}^{W/W_P}$; here the Kronecker function

$$\delta_{wW_P} : W/W_P \to \mathbb{Z}$$

corresponds to a cohomology class $[X^{wW_P}]$, and these form the dual basis to

$$\{[X_{wW_P}] \;:\; wW_P \in W/W_P\}.$$

In terms of these models, the Poincaré duality isomorphism is given by left multiplication by w_0 (see e.g. [BGG73]),

$$\begin{aligned} PD : H_k(G/P) &\to H^{2n-k}(G/P) \\ [X_{wW_P}] &\mapsto [X^{w_0 wW_P}] \end{aligned}$$

where $n = \dim_{\mathbb{C}} G/P$. Equivalently, the intersection pairing

$$\langle \cdot, \cdot \rangle : H_*(G/P) \times H_*(G/P) \to \mathbb{Z}$$

is given by

$$\langle [X_{wW_P}], [X_{w'W_P}] \rangle = \begin{cases} 1 & \text{if } w^{-1} w_0 w' \in W_P \\ 0 & \text{otherwise .} \end{cases}$$

3.5 Relative position

In this subsection, we give a more algebraic exposition of [KLP13, Section 3.3].

There is a combinatorial, W-valued invariant associated to a pair of points $p, q \in G/B$ called the *relative position* and denoted by $\mathrm{pos}(p, q)$. It can be defined as follows: Choose an element $g \in G$ such that $g \cdot p = eB$. Then $g \cdot q$ lies in the Schubert cell $C_w \subset G/B$ for a unique $w \in W$, and we define $\mathrm{pos}(p, q) = w$. One can check that this is independent of the choice of g.

To generalize this construction, let P and Q be standard parabolic subgroups of G corresponding to subsets $\Theta_P, \Theta_Q \subset \Delta$, so that in particular $B < P \cap Q$ and we have natural surjections $G/B \to G/P$ and $G/B \to G/Q$. Given $p \in G/P$ and $q \in G/Q$ we can select respective preimages $\widetilde{p}, \widetilde{q} \in G/B$ and consider their relative position $\mathrm{pos}(\widetilde{p}, \widetilde{q}) \in W$. While this element will depend on the choices of preimages, its double coset in $W_P \backslash W / W_Q$ depends only on p and q; we therefore define the relative position of p and q by

$$\mathrm{pos}_{P,Q}(p, q) = W_P \left(\mathrm{pos}(\widetilde{p}, \widetilde{q})\right) W_Q \in W_P \backslash W / W_Q.$$

Our previous definition is the special case $\mathrm{pos}_{B,B} = \mathrm{pos}$. It is immediate from the definition that the relative position is G-invariant in the sense that

$$\mathrm{pos}_{P,Q}(p, q) = \mathrm{pos}_{P,Q}(g(p), g(q)) \tag{3.3}$$

for all $g \in G$. Moreover, from its construction the relative position function is closely tied to the decompositions of G/P and G/Q into Schubert cells. We summarize its key properties in the following proposition, which follows easily from Theorem 3.1:

Proposition 3.2. *Suppose $p \in G/P$, $q \in G/Q$, and $g \in G$ satisfies $g \cdot p = eP$. Then we have $\mathrm{pos}_{P,Q}(p, q) = W_P w W_Q$ if and only if $g \cdot q$ is contained in the P-orbit on G/Q which is labeled by the double coset $W_P w W_Q$ in the sense of Theorem 3.1.(ii). Thus the level set $\{q \in G/Q \,:\, \mathrm{pos}_{P,Q}(p, q) = W_P w W_Q\}$ is a gPg^{-1}-orbit on G/Q. Moreover, the closure of this gPg^{-1}-orbit is given by the sublevel set*

$$\{q' \in G/Q \,:\, \mathrm{pos}_{P,Q}(p, q') \leqslant \mathrm{pos}_{P,Q}(p, q)\}$$

where $\leqslant$ is the Chevalley-Bruhat order on $W_P \backslash W / W_Q$.

In particular the Schubert cell in G/Q labeled by coset wW_Q is given by the level set

$$C_{wW_Q} = \{q \,:\, \mathrm{pos}_{B,Q}(eB, q) = wW_Q\}$$

and the corresponding Schubert variety X_{wW_Q} is the sublevel set

$$X_{wW_Q} = \overline{C_{wW_Q}} = \{q \,:\, \mathrm{pos}_{B,Q}(eB, q) \leqslant wW_Q\}.$$

□

This proposition shows that the ideal I in the Weyl group W, which corresponds to a closed union of Schubert varieties, equally corresponds to a union of sublevel sets of the relative position function over the generators of the ideal.

3.6 Parabolic pairs and thickenings

We have considered pairs of standard parabolic subgroups (P, Q) and the corresponding $W_P\backslash W/W_Q$-valued relative position function.

Now fix such a pair (P_A, P_D) of parabolics with P_A symmetric, and consider P_A-Anosov representations $\varrho : \pi \to G$. (Recall that by Proposition 2.5 there is no loss of generality in requiring P_A to be symmetric.) We consider the action of π on the partial flag variety G/P_D induced by ϱ, with the goal of finding a domain $\Omega \subset G/P_D$ on which the action is properly discontinuous. Thus the notation for the parabolics signifies that P_A is the "Anosov parabolic", while P_D is the "domain parabolic".

We make corresponding abbreviations $W_A := W_{P_A}$ and $W_D = W_{P_D}$ for the Weyl groups, and abbreviate the relative position function pos_{P_A,P_D} by $\mathrm{pos}_{A,D}$.

We say that an ideal $I \subset W$ has *type* (P_A, P_D) if I is left W_A-invariant and right W_D-invariant. Equivalently I is a union of double cosets $W_A w W_D$. Let $I \subset W$ be such an ideal. We can define the associated union of P_A-orbits

$$\Phi^I := \bigcup_{W_A w W_D \in W_A\backslash I/W_D} P_A w P_D \subset G/P_D$$

which we call the *model thickening* associated to I. (In [KLP13, Section 3.4.2] this is called a *thickening at infinity*.) By Theorem 3.1 the set Φ^I is a union of Schubert cells, and since I is an ideal, the set Φ^I is in fact a finite union of Schubert varieties. In particular it is a closed set.

Next, let $p \in G/P_A$ be a point of the flag variety. The *thickening of* p associated with I is a subset of G/P_D which we denote by Φ^I_p; it is defined using the relative position map:

$$\Phi^I_p := \{q \in G/P_D \ : \ \mathrm{pos}_{A,D}(p, q) \in W_A\backslash I/W_D\} \subset G/P_D.$$

This set is in fact a G-translate of the model thickening Φ^I:

Proposition 3.3. *For all $g \in G$ and $p \in G/P_A$, the thickenings satisfy*

$$\Phi^I_{g(p)} = g(\Phi^I_p).$$

In particular, if $p = gP_A \in G/P_A$, then $g(\Phi^I) = \Phi^I_p$.

Proof. By the G-invariance of the relative position function (3.3) we have

$$\mathrm{pos}_{A,D}(g(p), q) = \mathrm{pos}_{A,D}(p, g^{-1}(q)).$$

Therefore, $q \in \Phi^I_{g(p)}$ if and only if $g^{-1}(q) \in \Phi^I_p$, and if and only if $q \in g(\Phi^I_p)$. This proves $\Phi^I_{g(p)} = g(\Phi^I_p)$. The second statement follows immediately from the first and the fact that $\Phi^I = \Phi^I_{eP_D}$. $\square$

3.7 Limit sets and domains

Let P_A and P_D be parabolic subgroups, with P_A symmetric. For any subset $V \subset G/P_A$, define the thickening of V, denoted Φ^I_V, as the union of the thickenings of its points:

$$\Phi^I_V = \bigcup_{p \in V} \Phi^I_p$$

Let $\varrho : \pi \to G$ be a P_A-Anosov representation with limit curve $\xi : \partial_\infty \pi \to G/P_A$, and let I be an ideal of type (P_A, P_D). The *limit set* of ϱ relative to $I \subset W$ is defined as the thickening of the limit curve, i.e.

$$\Lambda^I_\varrho := \Phi^I_{\xi(\partial_\infty \pi)} = \bigcup_{t \in \partial_\infty \pi} \Phi^I_{\xi(t)} \subset G/P_D$$

The complement

$$\Omega^I_\varrho := G/P_D - \Lambda^I_\varrho$$

is the associated *domain*, which by the equivariance of ξ is a $\varrho(\pi)$-invariant open set. Let $\Gamma := \varrho(\pi)$.

The paramount result of [KLP13] establishes that if I is balanced, then the complement of the limit set furnishes a cocompact domain of proper discontinuity for the action of Γ on G/P_D. More generally:

Theorem 3.4 ([KLP13])**.**

(i) *If I is a slim, then the action of Γ on Ω^I_ϱ is properly discontinuous.*

(ii) *If I is fat, then the action of Γ on Ω^I_ϱ is cocompact.* □

In this construction, there remains the question of whether the domain Ω^I_ϱ could be empty. In [KLP13] and [GW12], various conditions are obtained ensuring the nonemptiness of the domains. In our primary applications, we will show that the corresponding domains are nonempty.

Regarding the structure of the limit set, the same authors show:

Theorem 3.5 ([KLP13, Lemma 3.38 and Lemma 7.4])**.** *If I is a slim ideal of type (P_A, P_D), then the set Λ^I_ϱ is a locally trivial bundle over $\partial_\infty \pi$ with typical fiber Φ^I.*

More generally, if $V \subset G/P_A$ is a compact set consisting of pairwise opposite points, then the set Φ^I_V is a locally trivial fiber bundle over V, where the projection $p : \Phi^I_V \to V$ is given by $p(\Phi^I_x) = x$. In particular, the thickenings $\{\Phi^I_x : x \in V\}$ are pairwise disjoint.

It will be important in what follows to know that this bundle is trivial for G-Fuchsian representations (which, we recall, are defined when G is simple and of adjoint type). This follows from similar considerations as those used in the proof of the Theorem above.

Lemma 3.6. *Let G be a complex simple Lie group of adjoint type. If $\varrho : \pi_1 S \to G$ is G-Fuchsian and I is a slim ideal of type (P_A, P_D), then there is a homeomorphism $\Lambda^I_\varrho \simeq \Phi^I \times S^1$.*

Proof. Recall that a bundle over S^1 that extends over the closed 2-disk is trivial. We show that Λ^I_ϱ admits such an extension.

By Corollary 2.2, the entire principal curve $f_G(\mathbb{P}^1_{\mathbb{C}}) := X \subset G/P_A$ consists of pairwise opposite points, hence by Theorem 3.5, the set Φ^I_X is a fiber bundle over X. By Theorem 2.9, the limit curve of a G-Fuchsian representation is the image of the limit curve of the associated Fuchsian group, which is simply the extended real line in the principal curve:

$$\xi(\partial_\infty \pi_1 S) = f_G(\mathbb{P}^1_{\mathbb{R}}) \subset G/P_A$$

Denoting the image as $f_G(\mathbb{P}^1_{\mathbb{R}}) := X_{\mathbb{R}} \subset X$, the limit set Λ^I_ϱ is

$$\Lambda^I_\varrho = p^{-1}(X_{\mathbb{R}}) \subset \Phi^I_X$$

where

$$p : \Phi^I_X \to X$$

is the aforementioned projection.

We have therefore described the bundle Λ^I_ϱ over base $S^1 \simeq \mathbb{P}^1_{\mathbb{R}} \simeq X_{\mathbb{R}}$ as the restriction to the equator of a bundle over $S^2 \simeq \mathbb{P}^1_{\mathbb{C}} \simeq X$. Since S^1 bounds a disk in $\mathbb{P}^1_{\mathbb{C}}$, the Lemma follows. $\square$

For later use, we record that the domains constructed in Theorem 3.4 for a G-Fuchsian representation are invariant under the full group $\iota_G(\mathrm{PSL}_2\mathbb{R})$.

Proposition 3.7. *Let G be a complex simple Lie group of adjoint type and $I \subset W$ an ideal of type (B, P_D). If $\varrho : \pi_1 S \to G$ is a G-Fuchsian representation, then the domain $\Omega^I_\varrho \subset G/P_D$ is invariant under $\iota_G(\mathrm{PSL}_2\mathbb{R})$.*

Proof. Since the limit curve $\xi(\partial_\infty \pi_1 S) = f_G(\mathbb{P}^1_{\mathbb{R}})$ in this case is an orbit of $\iota_G(\mathrm{PSL}_2\mathbb{R})$ on G/P_A, this is immediate from Proposition 3.3. $\square$

4 Size of the limit set

We now consider combinatorial properties of Weyl ideals and apply them to estimate the Hausdorff dimension of the limit sets described above. The results of this section are not used in Section 5, however they are essential to the complex geometry results of Section 6.

4.1 Weyl ideal combinatorics

As before we refer the reader to [Bou02] or [BB05] for more detailed discussion of the Coxeter group structure of the Weyl group W. We will also use the classification of complex simple Lie algebras into Cartan types A–G as described for example in [Bou02, Section VI.2].

As in the previous section we assume G is a complex semisimple Lie group, hence $\mathfrak{g}$ decomposes as a direct sum of simple Lie algebras, which we call the *simple factors*. There is a corresponding direct product decomposition of the Weyl group $W = W(G)$.

Our goal in this section is to show:

Theorem 4.1. *Let $I \subset W$ be a fat ideal.*

(i) *If G has no factors of type A_1, then I contains each element $w \in W$ with $\ell(w) \leqslant 1$.*

(ii) *If G has no factors of type A_1, A_2, A_3, or B_2, then I contains each element $w \in W$ with $\ell(w) \leqslant 2$.*

Note that by the exceptional isomorphisms, this also excludes types B_1, C_1, C_2 and D_3. In terms of the classical matrix groups, representatives of the excluded types are given by $A_1 = \mathfrak{sl}_2\mathbb{C}$, $A_2 = \mathfrak{sl}_3\mathbb{C}$, $A_3 = \mathfrak{sl}_4\mathbb{C}$, $B_2 = \mathfrak{sp}_4\mathbb{C}$.

Toward the proof of the theorem, we introduce the following terminology: An element $x \in W$ will be called *small* if $x \leqslant w_0 x$, where $w_0 \in W$ is the longest element (as in Section 2.1).

Lemma 4.2. *If $I \subset W$ is a fat ideal and $x \in W$ is small, then $x \in I$.*

Proof. Suppose for contradiction that x is small, I is a fat ideal, but that $x \notin I$. Then $w_0 x \in w_0(W - I)$, and since I is fat we have $w_0(W - I) \subset I$, thus $w_0 x \in I$. Since x is small we have $x < w_0 x$, and I is an ideal, so we find $x \in I$, a contradiction. □

Theorem 4.1 will follow from showing that elements of W of small length (i.e. "short" elements) are small.

Recall that the Weyl group W has a distinguished generating set S consisting of the simple root reflections, and that the length function $\ell : W \to \mathbb{Z}^{\geqslant 0}$ maps w to the minimum length of a word in S representing w. A word in S representing w of length $\ell(w)$ will be called *reduced*.

We will need the following construction of reduced words representing the longest element w_0. Recall that the *Coxeter number* h is a positive integer associated to a complex simple Lie algebra (see e.g. [Bou02, Section V.6.1]).

Lemma 4.3 (Bourbaki [Bou02, pp. 150–151])**.** *Suppose G is simple and has Coxeter number h. Let $S = S' \sqcup S''$ be a partition such that both S' and S'' generate free abelian subgroups of W. Let a (resp. b) denote the product of the elements of S' (resp. S''). Then:*

(i) *If h is even, then $w_0 = (ab)^{\frac{h}{2}}$ is a reduced word.*

(ii) *If h is odd, then $w_0 = (ab)^{\frac{h-1}{2}}a$ is a reduced word.* □

Note that the order in the product a does not matter since elements of S' commute, and similarly for b. Partitions $S = S' \sqcup S''$ of the type considered here always exist, as a each Dynkin diagram admits a 2-coloring and non-adjacent vertices correspond to commuting simple root reflections.

Lemma 4.3 also gives reduced words for w_0 when G is semisimple, by taking a product $\prod_i w_0^{(i)}$ of words of type (i) or (ii) for the longest elements $w_0^{(i)}$ of the Weyl groups of the simple factors.

Given a word w in the generating set S, we say that z is a *subword* of w if z is the result of deleting zero or more letters from arbitrary positions within w. The following relations between reduced words, subwords, and length in W are satisfied.

Lemma 4.4.

(i) *If $s \in S$ and $x = s_1 \cdots s_r$ is a reduced word, then $\ell(xs) = \ell(x) \pm 1$*

(ii) *If $\ell(xs) = \ell(x) - 1$ then for some k we have that $xs = s_1 \cdots \widehat{s_k} \cdots s_r$, and s_k is conjugate to s.*

(iii) *We have $x < y$ if and only if there is a reduced word representing x that is a subword of a reduced word representing y.* □

Proofs of these standards facts about Coxeter groups can be found for example in [BB05, Corollaries 1.4.4 and 2.2.3]. Note that these properties are often stated in terms of left multiplication by a reflection; the version for right multiplication stated above is equivalent, however, since the inversion map $w \mapsto w^{-1}$ is an automorphism of the Chevalley-Bruhat order.

Combining the previous lemmas we can now establish the key combinatorial property that underlies Theorem 4.1:

Lemma 4.5.

(i) *If each simple factor of G has Coxeter number at least 3, then each element of S is small.*

(ii) *If each simple factor of G has Coxeter number at least 5, then for any $s, t \in S$ the element $st \in W$ is small.*

Proof. First suppose G is simple with Coxeter number $h \geqslant 3$ and let $s \in S$. Note that $\ell(w_0 s) = \ell(w_0) - 1$ by (3.1). Apply Lemma 4.3 to a partition of S with $s \in S'$ to obtain a reduced expression of the form $w_0 = abaz$, where z is a (possibly empty) alternating product of a and b. The simple root reflection s appears at least twice in this word (once in each copy of a), hence by Lemma 4.4(ii) we find that s appears at least once in a reduced expression for $w_0 s$. This shows s is a subword of $w_0 s$, hence by Lemma 4.4(iii) we find $s < w_0 s$ and s is small.

Now suppose G is simple with Coxeter number $h \geqslant 6$. (The case $h = 5$ is considered separately below.) Let $s, t \in S$. We will show st is small. If $s = t$ then $s^2 = e$ and this is trivial, so we assume $s \neq t$. Then $\ell(st) = 2$, $\ell(stw_0) = \ell(w_0) - 2$, and $\ell(tw_0) = \ell(w_0) - 1$. Proceeding as before and using $h \geqslant 6$ we obtain a reduced expression $w_0 = abababz$, where we can assume s appears in product a. Applying Lemma 4.4(ii) twice we find that a reduced word for $w_0 st$ can be obtained from this one for w_0 by deleting two letters, and each such deletion may alter one of the copies of a or b in this word. However, this leaves at least one unaltered copy a to the left of an unaltered copy of b. That is, ab is a subword of a reduced expression for stw_0.

The simple root reflection t appears in either a or b. If it appears in b, then st is evidently a subword of ab. If t appears in a, then s and t commute and one of the equivalent words $st = ts$ is a subword of ab. In both cases we conclude that a reduced expression for st is a subword of a reduced expression for stw_0, and by Lemma 4.4(iii) we have $st < w_0 st$. That is, st is small.

If G is simple of $h = 5$ then G is of type A_4, hence $W \simeq S_5$. In this case it can be checked directly that the nine nontrivial elements which are products of pairs of simple root reflections are small. We omit the details of this verification.

Finally suppose G is semisimple. We have a reduced expression for w_0 that is a product over the simple factors. If each simple factor has Coxeter number at least 3, we find as before that the reduced expression for w_0 can be constructed to use a given simple root reflection s at least twice, and hence that s is small. If each simple factor has Coxeter number at least 5, and if s, t are simple root reflections ($s \neq t$), then a reduced word for $w_0 st$ is obtained by deleting two letters from the word for w_0, and the deleted letters are respective conjugates of s and t. If s and t lie in the same simple factor of W, then the deleted letters are both in the corresponding factor of w_0, and the argument above in the simple case shows that st is a subword of the result. If s and t lie in distinct simple factors (and hence commute), we recall that each can be assumed to appear at least twice in its factor and hence each appears at least once after the deletion. Thus $st = ts$ is also a subword of a reduced expression for $w_0 st$ in this case. We have therefore shown st is small. □

Using this lemma, the proof of Theorem 4.1 is straightforward:

Proof of Theorem 4.1. The elements $x \in W$ with $\ell(x) \leqslant 1$ are the simple root reflections and the identity element. The only simple Lie algebra of Coxeter number less than 3 is A_1, hence if G has no simple factors of this type then Lemma 4.5(i) shows that the simple root reflections are small. The identity element is also small. By Lemma 4.2 we find that these elements lie in any fat ideal $I \subset W$, and part (i) of the theorem follows.

In exactly the same way, part (ii) follows from Lemma 4.5(ii) because the only simple lie algebras with Coxeter number less than 5 are A_1, A_2, A_3, and B_2, and the elements $x \in W$ with $\ell(x) \leqslant 2$ are the identity element, the simple root reflections, and products of pairs of simple root reflections, all of which are small in this case. □

4.2 Hausdorff dimension of limit sets

Now we will bound the Hausdorff dimension of the limit set of an Anosov representation in terms of the Hausdorff dimension of its limit curve and the combinatorial size of the ideal defining the thickening.

All of the sets for which we discuss dimension are closed subsets of compact manifolds. When regarding such sets as metric spaces (for example when computing dimensions) we always consider them to be equipped with the distance obtained by restricting the distance induced by an arbitrary Riemannian metric on the ambient manifold. Since any two Riemannian metrics on a compact manifold are bi-Lipschitz, our results will not depend on the particular metric chosen.

Let $P_A < G$ be a symmetric parabolic subgroup of a complex semisimple Lie group G. Let $V \subset G/P_A$ be a closed subset consisting of pairwise opposite points. The property of a pair of points being opposite is an open condition since it coincides with the unique open orbit of G acting diagonally on $G/P_A \times G/P_A$. (Here we are using the fact that P_A is symmetric so that it is conjugate to any of its opposite parabolic subgroups.)

Let W be the Weyl group of G. We begin with the following general fact, which is a straightforward generalization of Theorem 3.5:

Proposition 4.6. *Let $P_D < G$ be a parabolic subgroup and let $I \subset W$ be a slim ideal of type (P_A, P_D). Let $V \subset G/P_A$ denote a compact subset consisting of pairwise opposite points. Then the fiber bundle $p : \Phi^I_V \to V$ admits Lipschitz local parameterizations; that is, each point $x \in V$ has a neighborhood U_x such that there exists a Lipschitz homeomorphism*

$$U_x \times \Phi^I \to p^{-1}(U_x).$$

In fact, this proposition follows easily from the proofs of [KLP13, Lemmas 3.39 and Lemma 7.4] which we stated as Theorem 3.5 above. We will simply recall enough of the construction used by those authors to make the Lipschitz property evident.

Proof. Note that the set Φ^I is compact. For $x \in V$ let U_x be a relatively compact neighborhood of x in V over which there exists a smooth section $s : U_x \to G$ of the quotient map $G \to G/P_A$, and choose such a section. In the proof of [KLP13, Lemma 7.4] it is shown that the map

$$\begin{aligned} U_x \times \Phi^I &\to p^{-1}(U_x) = \Phi^I_{U_x} \\ (x, y) &\mapsto s(x)(y) \end{aligned}$$

gives a local trivialization of the bundle $\Phi^I_V \to V$. However, as it is the restriction of the smooth action map $G \times G/P_D \to G/P_D$ to the relatively compact set $s(U_x) \times \Phi^I_V$, this map is also Lipschitz. $\square$

We now come to the main result of this section.

Theorem 4.7. *Let $P_A, P_D < G$ be a pair of parabolic subgroups with P_A symmetric. Let $\varrho : \pi \to G$ be a P_A-Anosov representation of a word hyperbolic group with limit curve $\xi : \partial_\infty \pi \to G/P_A$. Let $I \subset W$ be a slim ideal of type (P_A, P_D). Then the limit set $\Lambda^I_\varrho \subset G/P_D$ satisfies*

$$\dim_{\mathrm{H}}(\Lambda^I_\varrho) \leqslant \dim_{\mathrm{H}}\left(\xi(\partial_\infty \pi)\right) + 2 \max_{w \in I/W_D} \ell(w).$$

Here, the Hausdorff dimension is computed with respect to any Riemannian metrics on G/P_A and G/P_D, and ℓ denotes the length function associated to the Chevalley-Bruhat order on W/W_D.

Proof. Recall $\Lambda^I_\varrho = \Phi^I_{\xi(\partial_\infty \pi)}$ and $\xi(\partial_\infty \pi)$ is a compact set consisting of pairwise opposite points (by Theorem 3.5). Applying Proposition 4.6 we obtain a finite open cover $\{U_i\}$ of $\partial_\infty \pi$ by sets whose images by ξ are trivializing open sets for the bundle Λ^I_ϱ, and over which this bundle has Lipschitz parameterizations. Since Lipschitz maps do not increase Hausdorff dimension, and since Hausdorff dimension is finitely stable, we find

$$\dim_{\mathrm{H}}(\Lambda^I_\varrho) \leqslant \max_i \dim_{\mathrm{H}}(\xi(U_i) \times \Phi^I). \tag{4.1}$$

On the other hand, the Hausdorff dimension of a product can be bounded in terms of the Hausdorff dimension and upper Minkowski dimension (also known as upper box counting dimension) of the factors [Fal14, Formula 7.3]:

$$\dim_{\mathrm{H}}(\xi(U_i) \times \Phi^I) \leqslant \dim_{\mathrm{H}}(\xi(U_i)) + \overline{\dim}_{\mathrm{M}}(\Phi^I)$$

However, Φ^I has a finite stratification by manifolds (the Schubert cells corresponding to elements of I), and hence its upper Minkowski dimension is equal to the maximum real dimension of these manifolds (see e.g. [Fal14, Section 3.2]), which is $2 \max_{w \in I/W_D} \ell(w)$. Also, since $\xi(U_i)$ is a subset of $\xi(\partial_\infty \pi)$ we have $\dim_{\mathrm{H}}(\xi(U_i)) \leqslant \dim_{\mathrm{H}}(\xi(\partial_\infty \pi))$. We conclude

$$\dim_{\mathrm{H}}(\xi(U_i) \times \Phi^I) \leqslant \dim_{\mathrm{H}}\left(\xi(\partial_\infty \pi)\right) + 2 \max_{w \in I/W_D} \ell(w).$$

Substituting this bound into (4.1), the Theorem follows. $\square$

We note that in case the right hand side of the bound from Theorem 4.7 is less than the real dimension of G/P_D itself, it follows that the limit set has positive "Hausdorff codimension" and that Ω^I_ϱ is nonempty. We state the resulting criterion separately:

Theorem 4.8. *Let $\varrho : \pi \to G$ be a P_A-Anosov representation for a symmetric parabolic subgroup $P_A < G$. Suppose $I \subset W$ is a balanced ideal of type (P_A, P_D) with limit curve $\xi : \partial_\infty \pi \to G/P_A$. Let $n = \dim_{\mathbb{C}} G/P_D$. If*

$$\dim_{\mathrm{H}} \xi(\partial_\infty \pi) < 2\left(n - \max_{w \in I/W_D} \ell(w)\right)$$

then the corresponding domain $\Omega_\varrho^I \subset G/P_D$ is nonempty. In particular:

(i) *If G is not isomorphic to* $\mathrm{PSL}_2\mathbb{C}$, *then the domain is nonempty for any balanced ideal provided* $\dim_{\mathrm{H}} \xi(\partial_\infty \pi) < 2$, *and*

(ii) *If G is not isomorphic to types A_1, A_2, A_3 or B_2, then the domain is nonempty for any balanced ideal provided* $\dim_{\mathrm{H}} \xi(\partial_\infty \pi) < 4$.

Proof. By Theorem 4.7, the first inequality in the statement of the Theorem implies that $\dim_{\mathrm{H}}(\Lambda_\varrho^I) < \dim_{\mathrm{H}}(G/P_D)$, and hence that Λ_ϱ^I is a proper subset. Equivalently Ω_ϱ^I is nonempty.

Next, claim (i) follows from Theorem 4.1 since this implies that $\max_{w \in I/W_D} \ell(w) < n-1$ and claim (ii) also follows from Theorem 4.1 since the exclusion of types A_1, A_2, A_3 and B_2 implies $\max_{w \in I/W_D} \ell(w) < n - 2$. □

Our main application of this result will be to estimate the Hausdorff dimension of limit sets for quasi-Fuchsian groups. We find:

Theorem 4.9. *Let G be a complex simple Lie group of adjoint type and rank at least two with Weyl group W. Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation and $I \subset W$ a balanced ideal of type (B, P_D). Let n denote the complex dimension of G/P_D. Then the limit set $\Lambda_\varrho^I \subset G/P_D$ satisfies*

$$m_{2n-2}(\Lambda_\varrho^I) = 0.$$

Furthermore, if G is not of type A_2, A_3 or B_2, then

$$m_{2n-4}(\Lambda_\varrho^I) = 0.$$

Here m_k denotes the k-dimensional Hausdorff measure associated to any Riemannian metric on G/P_D.

Proof. By Theorem 4.1, the hypotheses imply $\max_{w \in I/W_D} \ell(w) \leqslant n - 2$. As the limit curve of a quasi-Fuchsian group is a quasi-circle in $\mathbb{P}^1_{\mathbb{C}}$, its Hausdorff dimension is strictly less than 2. By Theorem 2.9, the limit curve of a G-quasi-Fuchsian group is the image of such a quasi-circle by the smooth embedding $f_G : \mathbb{P}^1_{\mathbb{C}} \to G/P_D$, hence $\xi(\partial_\infty \pi_1 S)$ also has Hausdorff dimension less than 2. Applying Theorem 4.7 gives

$$\dim_{\mathrm{H}}(\Lambda_\varrho^I) < 2 + 2(n-2) = 2n - 2.$$

and thus $m_{2n-2}(\Lambda^I_\varrho) = 0$.

If we furthermore exclude types A_2, A_3 and B_2, then Theorem 4.1 gives $\max_{w \in I/W_D} \ell(w) \leqslant n-3$, and proceeding as above we find $m_{2n-4}(\Lambda^I_\varrho) = 0$. □

We note that, in particular, the domains in these cases considered in Theorem 4.1 are always nonempty.

5 Topology

We now begin one of our central investigations of the paper—studying the topology of the domains and quotient manifolds for G-quasi-Hitchin representations. We do this by first reducing to the G-Fuchsian case (in Sections 5.1–5.2) and then studying the Fuchsian case in Sections 5.3–5.5.

5.1 Anosov components

Let π be a finitely generated group and G a complex semisimple Lie group. By choosing a finite generating set of π, the set $\mathrm{Hom}(\pi, G)$ can be identified with a complex affine subvariety of G^N for some $N \in \mathbb{N}$. Thus $\mathrm{Hom}(\pi, G)$ has both the Zariski topology and the compact-open topology of maps from the discrete space π to the manifold G, the latter of which we will call the analytic topology. Throughout this section, we use *component* to mean a connected component of a set with respect to the analytic topology.

Let P_A be a symmetric parabolic subgroup of G. Given a P_A-Anosov representation $\varrho : \pi \to G$, let $\mathcal{A}(\varrho, P_A) \subset \mathrm{Hom}(\pi, G)$ denote the connected component of the set of P_A-Anosov representations that contains ϱ. We call $\mathcal{A}(\varrho, P_A)$ the *Anosov component* of ϱ.

For example, the quasi-Hitchin set $\widetilde{\mathfrak{QH}}(S, G, P_A)$ for a complex simple adjoint group G, as defined in Section 2.7, is equivalently described as the Anosov component $\mathcal{A}(\varrho, P_A)$ of any G-Fuchsian representation $\varrho : \pi_1 S \to G$.

Since $\mathrm{Hom}(\pi, G)$ is a complex affine variety, it has a Zariski-open subset $\mathrm{Hom}^{\mathrm{sm}}(\pi, G)$ of smooth points, each component of which is a complex manifold. We call this the *smooth locus*.

If $\varrho \in \mathrm{Hom}^{\mathrm{sm}}(\pi, G)$ is P_A-Anosov, we define $\mathcal{A}_{\mathrm{sm}}(\varrho, P_A)$ to be the component of ϱ in $\mathcal{A}(\varrho, P_A) \cap \mathrm{Hom}^{\mathrm{sm}}(\pi, G)$; thus $\mathcal{A}_{\mathrm{sm}}(\varrho, P_A)$ is a connected complex manifold, and we refer to it as the *smooth component* of ϱ.

Extending the notation of Section 2.7, we write $\widetilde{\mathfrak{QH}}_{\mathrm{sm}}(S, G, P_A) := \mathcal{A}_{\mathrm{sm}}(\varrho, P_A)$ for the smooth component associated to a G-Fuchsian representation $\varrho : \pi_1 S \to G$.

5.2 Universal domains

Let P_D be another parabolic subgroup of G, and let $I \subset W$ be a balanced ideal of type (P_A, P_D). Let $\varrho_0 : \pi \to G$ be a P_A-Anosov representation, and write $\mathcal{A} = \mathcal{A}(\varrho_0, P_A)$ for its

Anosov component. Each representation in $\varrho \in \mathcal{A}$ has an associated domain $\Omega_\varrho^I \subset G/P_D$; assembling all of these into a single object, we define the *universal domain* $\widetilde{\mathcal{V}}^I$ by

$$\widetilde{\mathcal{V}}^I := \{(\varrho, x) \in \mathcal{A}(\varrho, P_A) \times (G/P_D) \ : \ x \in \Omega_\varrho^I\}.$$

which by Proposition 2.6(iv) is an open subset of $\mathcal{A} \times G/P_D$. Let $\widetilde{\Pi} : \widetilde{V}^I \to \mathcal{A}$ denote the projection on to the first factor, so that $\widetilde{\Pi}^{-1}(\varrho) = \{\varrho\} \times \Omega_\varrho^I$.

The group π acts on $\widetilde{\mathcal{V}}^I$ by

$$\gamma \cdot (\varrho, x) = (\varrho, \varrho(\gamma)(x)).$$

Let $\mathcal{V}^I := \widetilde{\mathcal{V}}^I/\pi$ denote the quotient by this action. Since $\widetilde{\Pi}(\gamma \cdot (\varrho, x)) = \widetilde{\Pi}(\varrho, x) = \varrho$, there is an induced map $\Pi : \mathcal{V}^I \to \mathcal{A}$ such that $\Pi^{-1}(\varrho) = \{\varrho\} \times \mathcal{W}_\varrho^I$; in this way, $\mathcal{V}^I$ is the *universal quotient manifold* over $\mathcal{A}$.

When $\varrho_0 \in \mathrm{Hom}^{\mathrm{sm}}(\pi, G)$, denote by $\widetilde{\mathcal{V}}_{\mathrm{sm}}$ and $\mathcal{V}_{\mathrm{sm}}$ the associated families over $\mathcal{A}_{\mathrm{sm}} = \mathcal{A}_{\mathrm{sm}}(\varrho_0, P_A)$, i.e. $\widetilde{\mathcal{V}}_{\mathrm{sm}} = \widetilde{\Pi}^{-1}(\mathcal{A}_{\mathrm{sm}})$, $\mathcal{V}_{\mathrm{sm}} = \Pi^{-1}(\mathcal{A}_{\mathrm{sm}}) = \widetilde{\mathcal{V}}_{\mathrm{sm}}/\pi$.

Lemma 5.1. *The map $\Pi : \mathcal{V}^I \to \mathcal{A}$ is continuous and proper. Furthermore, if $\varrho_0 \in \mathrm{Hom}^{\mathrm{sm}}(\pi, G)$, then $\mathcal{V}^I_{\mathrm{sm}}$ is a complex manifold and the restricted map $\Pi : \mathcal{V}^I_{\mathrm{sm}} \to \mathcal{A}_{\mathrm{sm}}$ is a holomorphic submersion.*

Proof. The continuity and properness of Π follow immediately from the definitions and compactness of the fibers $\Pi^{-1}(\varrho) \simeq \mathcal{W}_\varrho^I$. By Proposition 2.6(vi), $\widetilde{\mathcal{V}}^I_{\mathrm{sm}}$ is an open set in the complex manifold $\mathcal{A}_{\mathrm{sm}} \times G/P_D$ and is therefore a complex manifold. Furthermore π acts smoothly on $\widetilde{\mathcal{V}}^I_{\mathrm{sm}}$ and $\widetilde{\mathcal{V}}^I_{\mathrm{sm}} \to \mathcal{V}^I_{\mathrm{sm}}$ is a covering map, so $\mathcal{V}^I_{\mathrm{sm}}$ is a complex manifold.

Finally, the map $\widetilde{\Pi}\Big|_{\widetilde{\mathcal{V}}^I_{\mathrm{sm}}}$ is a holomorphic submersion since it is the restriction of the projection of the product (complex) manifold $\mathcal{A}_{\mathrm{sm}} \times G/P_D$ onto its first factor. Since $\Pi|_{\mathcal{V}^I_{\mathrm{sm}}}$ is covered by $\widetilde{\Pi}\Big|_{\widetilde{\mathcal{V}}^I_{\mathrm{sm}}}$, it is a submersion as well. ☐

In the terminology of Kodaira-Spencer [KS58], Lemma 5.1 shows that $\mathcal{V}^I_{\mathrm{sm}}$ constitutes a family of compact complex manifolds with base $\mathcal{A}_{\mathrm{sm}}$.

By Ehresmann's lemma [Ehr95], the previous Lemma shows that the universal quotient manifold is actually a bundle over $\mathcal{A}_{\mathrm{sm}}$, and we obtain:

Corollary 5.2. *The family $\Pi : \mathcal{V}^I_{\mathrm{sm}} \to \mathcal{A}_{\mathrm{sm}}$ is a smooth locally trivial fiber bundle.*

In particular, for any balanced ideal I of type (P_A, P_D) and any pair $\varrho, \varrho' \in \mathcal{A}_{\mathrm{sm}}$, the quotient manifolds $\mathcal{W}^I_\varrho$ and $\mathcal{W}^I_{\varrho'}$ are diffeomorphic. ☐

We remark that in [GW12] it was shown that the homeomorphism type is constant on Anosov components for the quotients of the domains of discontinuity constructed by those authors. The argument given there is quite general, however, and would also apply in the present situation. We have given the detailed argument above in order to emphasize the smoothness of the resulting map.

Applying Corollary 5.2 to a G-Fuchsian representation (assuming now that G is complex simple and adjoint), we have:

Theorem 5.3. *The space $\widetilde{\mathfrak{QH}}(S, G, P)_{\mathrm{sm}}$ is nonempty and contains all G-Hitchin representations. The diffeomorphism type of the quotient manifold $\mathcal{W}^I_\varrho$ is independent of the homomorphism $\varrho \in \widetilde{\mathfrak{QH}}(S, G, P)_{\mathrm{sm}}$ and in particular each such quotient manifold is diffeomorphic to one obtained from a G-Fuchsian representation.*

Proof. By Theorem 2.8 the set $\widetilde{\mathfrak{QH}}(S, G, P)_{\mathrm{sm}}$ contains the G-Hitchin representations; the rest of the statement is the result of specializing Corollary 5.2 to the case $\mathcal{A}_{\mathrm{sm}} = \widetilde{\mathfrak{QH}}(S, G, P)_{\mathrm{sm}}$. □

5.3 Homology and cohomology of thickenings

Starting toward our study of the topology of G-Fuchsian quotient manifolds associated to a Chevalley-Bruhat ideal I, we begin by considering the topology of the model thickening $\Phi^I \subset G/P_D$.

Lemma 5.4. *Let $I \subset W$ be a right W_D-invariant ideal. Then in the Schubert cell basis for $H_*(G/P_D)$, the map*

$$i : H_*(\Phi^I) \to H_*(G/P_D)$$

induced by the inclusion $\Phi^I \hookrightarrow G/P_D$ corresponds to the natural embedding of free $\mathbb{Z}$-modules

$$\mathbb{Z}[I/W_D] \hookrightarrow \mathbb{Z}[W/W_D].$$

Proof. The model thickening Φ^I is a closed set that is a union of Schubert cells, hence it a subcomplex of the cell structure on G/P_D. Using the labeling of cells by W_D-cosets, the natural map $\mathbb{Z}[I/W_D] \hookrightarrow \mathbb{Z}[W/W_D]$ becomes the map on cellular chain complexes induced by the inclusion of Φ^I. Since the boundary maps of these chain complexes vanish identically (as there are no odd-dimensional cells), this is naturally isomorphic to the induced map on homology. □

Taking duals, Lemma 5.4 identifies the cohomology pullback map associated to the inclusion $\Phi^I \hookrightarrow G/P_D$ with the natural surjective map $\mathbb{Z}^{W/W_D} \to \mathbb{Z}^{I/W_D}$.

Next, we show that the pair of orthogonal ideals $I, I^\perp$ corresponds naturally to a splitting of the homology $H_*(G/P_D)$ as a direct sum.

Lemma 5.5. *For each right W_D-invariant ideal I there is a split exact sequence*

$$0 \to H_*(\Phi^I) \xrightarrow{i} H_*(G/P_D) \to H^{2n-*}(\Phi^{I^\perp}) \to 0$$

where i is the map induced by $\Phi^I \hookrightarrow G/P_D$ and $n = \dim_{\mathbb{C}} G/P_D$.

Proof. Splitting is automatic since $H^{2n-*}(\Phi^{I^\perp})$ is free abelian (by the previous lemma). To construct the exact sequence, let $j : H_*(G/P_D) \to H^{2n-*}(\Phi^{I^\perp})$ denote the composition of the Poincaré duality map with the pullback map on cohomology from the inclusion $\Phi^{I^\perp} \to G/P_D$. As a composition of an isomorphism and a surjection (the latter using the previous lemma), we see j is itself surjective. Its kernel consists of classes that are orthogonal (with respect to the intersection pairing) to $H_{2n-*}(\Phi^{I^\perp})$. Identifying $H_{2n-*}(\Phi^{I^\perp})$ with the submodule $\mathbb{Z}[I^\perp/W_D]$ of $\mathbb{Z}[W/W_D]$, the description of the intersection pairing from Section 3.4 shows that this submodule pairs nontrivially with basis elements in $w_0 I^\perp$, and is zero otherwise. That is, the orthogonal is $\mathbb{Z}[(W - w_0 I^\perp)/W_D]$. Recalling that $I^\perp = w_0(W - I)$ and $w_0^2 = e$ we see that this is simply $\mathbb{Z}[I/W_D] \simeq i(H_*(\Phi^I))$ as required. □

We remark that this lemma essentially describes the (co)homologial consequence of the disjoint union decomposition $(W/W_D) = (I/W_D) \sqcup (w_0 I^\perp/W_D)$. In case I is slim the description of the intersection pairing on G/P_D from Section 3.4 shows that the image of $H_*(\Phi^I)$ is an isotropic space for this pairing (i.e. the restriction of the intersection form vanishes identically). Therefore, for a *balanced* ideal I the exact sequence of Lemma 5.5 represents an associated "Lagrangian splitting" of the homology $H_*(G/P_D)$.

5.4 Homology of domains of proper discontinuity

We now turn to the topology of domains Ω_ϱ^I.

Theorem 5.6. *Let G be a complex simple Lie group of adjoint type and let $\varrho \in \widetilde{\mathfrak{QH}}_{\mathrm{sm}}(S, G, P_A) \subset \mathrm{Hom}(\pi_1 S, G)$. If I is a slim ideal of type (P_A, P_D) with associated model thickening Φ^I and domain $\Omega_\varrho^I \subset G/P_D$, then there is a split short exact sequence*

$$0 \to H^{2n-2-k}(\Phi^I, \mathbb{Z}) \to H_k(\Omega_\varrho^I, \mathbb{Z}) \to H_k(\Phi^{I^\perp}, \mathbb{Z}) \to 0 \tag{5.1}$$

where $n = \dim_{\mathbb{C}} G/P_D$. In particular, the homology groups of Ω_ϱ^I are free abelian. In addition:

(i) *The odd homology groups of Ω_ϱ^I vanish,*

(ii) *If I is balanced, then the homology of Ω_ϱ^I satisfies*

$$H_k(\Omega_\varrho^I, \mathbb{Z}) \simeq H^{2n-2-k}(\Omega_\varrho^I, \mathbb{Z}).$$

Observe that when applied to a balanced ideal I, this theorem incorporates the results stated as Theorems B and C in the introduction, with the exception of statement (iii) of Theorem C.

In the proof, we will omit the $\mathbb{Z}$-coefficients to simplify notation.

Proof. By Theorem 5.3 it suffices to consider the case when ϱ is G-Fuchsian. Assume this from now on.

Poincaré-Alexander-Lefschetz duality yields a canonical isomorphism

$$H^{2n-j}(G/P_D, \Lambda_\varrho^I) \simeq H_j(\Omega_\varrho^I). \tag{5.2}$$

Note that there is no need to use Čech cohomology on the left here, since Λ_ϱ^I is a CW-complex: Theorem 3.6 shows that it is homeomorphic to the product of an algebraic variety and S^1.

Since the cohomology of G/P_D vanishes in odd degrees, the long exact sequence in cohomology of the pair $(G/P_D, \Lambda_\varrho^I)$ decomposes into five-term sequences centered on the even degree cohomology groups of G/P_D:

$$\begin{multline} 0 \to H^{2n-2j-1}(\Lambda_\varrho^I) \to H^{2n-2j}(G/P_D, \Lambda_\varrho^I) \to H^{2n-2j}(G/P_D) \xrightarrow{*} \\ \xrightarrow{*} H^{2n-2j}(\Lambda_\varrho^I) \to H^{2n-2j+1}(G/P_D, \Lambda_\varrho^I) \to 0. \end{multline} \tag{5.3}$$

Using Theorem 3.6, the Künneth Theorem implies $H^{2n-2j}(\Lambda_\varrho^I) \simeq H^{2n-2j}(\Phi^I)$. Post-composing with this isomorphism, the map labeled $(*)$ becomes the pullback map on cohomology of degree $(2n-2j)$ induced by the inclusion $\Phi^I \hookrightarrow G/P_D$. Taking the dual of the exact sequence from Lemma 5.5, we find that this map is surjective with kernel isomorphic to $H_{2j}(\Phi^{I^\perp})$.

By the surjectivity of $(*)$ and the Poincaré-Alexander-Lefschetz isomorphism (5.2), the exactness of (5.3) at the right implies that

$$0 = H^{2n-2j+1}(G/P_D, \Lambda_\varrho^I) \simeq H_{2j-1}(\Omega_\varrho^I)$$

which is statement (i) of the Theorem. Since the (co)homology of Φ^I and $\Phi^{I^\perp}$ vanish in odd degrees (by Lemma 5.4), this also trivially verifies the existence of the exact sequence (5.1) when the degree is odd.

For even degrees, since the map labeled by $(*)$ has kernel isomorphic to $H_{2j}(\Phi^{I^\perp})$, the five-term exact sequence restricts to a short exact sequence

$$0 \to H^{2n-2j-1}(\Lambda_\varrho^I) \to H^{2n-2j}(G/P_D, \Lambda_\varrho^I) \to H_{2j}(\Phi^{I^\perp}) \to 0. \tag{5.4}$$

The Künneth Theorem, Theorem 3.6, and the vanishing of the odd-dimensional cohomology of Φ^I imply $H^{2n-2j-1}(\Lambda_\varrho^I) \simeq H^{2n-2j-1}(\Phi^I \times S^1) \simeq H^{2n-2j-2}(\Phi^I)$. Using this isomorphism to replace the initial term in (5.4) and the Poincaré-Alexander-Lefschetz duality isomorphism (5.2) to replace the central term with $H_{2j}(\Omega_\varrho^I)$ yields the desired short exact sequence

$$0 \to H^{2n-2j-2}(\Phi^I) \to H_{2j}(\Omega_\varrho^I) \to H_{2j}(\Phi^{I^\perp}) \to 0.$$

Since $H_{2j}(\Phi^{I^\perp})$ is a free abelian group, the sequence splits.

Finally, statement (ii) follows immediately by taking the dual of the exact sequence (5.1) and applying the universal coefficients theorem. $\square$

As a corollary of this result, we find a simple formula for the Betti numbers of the domain of discontinuity, which we state only for the case when I is balanced. Note that Lemma 5.4 shows that $b_{2k}(\Phi)$ is the number of elements of I/W_D of length k. Thus if $I = I^{\perp}$, the Theorem above gives:

Corollary 5.7. *Under the hypotheses of Theorem 5.6, if I is a balanced ideal, then the Betti numbers of the domain of discontinuity in G/P_D are given by*

$$b_{2k}(\Omega_\varrho^I) = r_k + r_{n-1-k}$$

where r_k is the number of elements of I/W_D of length k and $n = \ell(w_0 W_D) = \dim_{\mathbb{C}} G/P_D$.
□

As this corollary is statement (iii) of Theorem C , we have now completed the proofs of Theorems B and C. Using the corollary above to calculate the Euler characteristic of Ω_ϱ^I, we also obtain:

Corollary 5.8. *Under the hypotheses of Theorem 5.6, if I is a balanced ideal, then the Euler characteristic of the domain of discontinuity is given by*

$$\chi(\Omega_\varrho^I) = \chi(G/P_D) = |W/W_D|.$$

Proof. Since Ω^I has only even-dimensional homology, the Euler characteristic is the sum of its Betti numbers. Using the formula of Corollary 5.7, each term r_k appears twice in this sum, hence $\chi(\Omega_\varrho^I) = 2|I/W_D|$. Since a balanced ideal satisfies $2|I| = |W|$, a balanced W_D-invariant ideal satisfies $2|I/W_D| = |W/W_D|$, and the desired formula for $\chi(\Omega_\varrho^I)$ follows. □

5.5 Homology of quotient manifolds

Next we show that Serre spectral sequence for the covering $\Omega_\varrho^I \to \mathcal{W}_\varrho^I$ degenerates, yielding:

Theorem 5.9. *Let G be a complex simple Lie group of adjoint type and $\varrho \in \widetilde{\mathfrak{QH}}_{\mathrm{sm}}(S, G, P_A)$ where $P_A < G$ is a symmetric parabolic subgroup.*

If I is a balanced ideal of type (P_A, P_D) with associated domain $\Omega_\varrho^I \subset G/P_D$, let $\mathcal{W}_\varrho^I$ denote the compact quotient manifold. Then, there isomorphism of graded $\mathbb{Z}$-modules

$$H_*(\mathcal{W}_\varrho^I, \mathbb{Z}) \simeq H_*(S, \mathbb{Z}) \otimes H_*(\Omega_\varrho^I, \mathbb{Z}).$$

As in Corollary 5.7, this shows $H_k(\mathcal{W}_\varrho^I, \mathbb{Z})$ is free abelian for each k and its rank is computable from the combinatorial data of the ideal I and the length function ℓ on W/W_D. Also, using Corollary 5.8 we obtain the result stated in the introduction as Corollary 1.2:

Corollary 5.10. *For $\mathcal{W}_\varrho^I$ as above we have $\chi(\mathcal{W}_\varrho^I) = \chi(S)\chi(G/P_D)$, and so in particular $\chi(\mathcal{W}_\varrho^I) = (2-2g)|W/W_D| < 0$ where $g \geqslant 2$ is the genus of S.* □

This corollary indicates the importance of the (co)homology calculation since we cannot distinguish the quotient manifolds for different choices of ideals $I \subset W$ using the Euler characteristic.

Proof of Theorem 5.9. As before, Theorem 5.3 reduces the statement to the case of G-Fuchsian ϱ. Let $E^2_{p,q} = H_p(S, H_q(\Omega^I_\varrho, \mathbb{Z}))$ denote the E^2 page of the Serre spectral sequence for homology of the regular covering $\Omega^I_\varrho \to \mathcal{W}^I_\varrho$. Since S is a $K(\pi_1 S, 1)$, there is an isomorphism

$$E^2_{p,q} \simeq H_p(\pi_1 S, H_q(\Omega^I_\varrho, \mathbb{Z})_\varrho)$$

where the right hand side is group homology, and where the $\pi_1 S$-action on $H_q(\Omega^I_\varrho, \mathbb{Z})$ is prescribed by ϱ. Furthermore, we claim

$$H_p(\pi_1 S, H_q(\Omega^I_\varrho, \mathbb{Z})_\varrho) \simeq H_p(S, \mathbb{Z}) \otimes H_q(\Omega^I_\varrho, \mathbb{Z}). \tag{5.5}$$

which follows if we show $H_*(\Omega^I_\varrho, \mathbb{Z})$ is a trivial $\pi_1 S$-module. However, by Proposition 3.7 the domain Ω^I_ϱ associated to a G-Fuchsian representation is invariant under the action of the real principal three-dimensional subgroup $\iota_G(\mathrm{PSL}_2\mathbb{R}) := \mathfrak{S}_\mathbb{R}$ on G/P_D. Since $\mathfrak{S}_\mathbb{R}$ is a connected Lie group, the action of any element of this group on Ω^I_ϱ is homotopic to the identity and hence acts trivially on $H_*(\Omega^I_\varrho, \mathbb{Z})$. Since $\varrho(\pi_1 S) \subset \mathfrak{S}_\mathbb{R}$, this gives the desired triviality of the $\pi_1 S$-module $H_*(\Omega^I_\varrho, \mathbb{Z})$.

Next, we claim that the spectral sequence degenerates at the E^2-page. First, from (5.5) we find $E^2_{p,q} = 0$ if $p > 2$ (since S has real dimension 2) or if q is odd (by vanishing of odd homology of Ω^I_ϱ). The condition on p leaves the E^2-differentials $\partial^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$ as the only potentially nontrivial maps, however these change the parity of q and hence either the domain or codomain is trivial. Thus all differentials vanish at the E^2-page.

Finally, since all groups on the E^2-page are free abelian (which follows from the homology of both Ω^I_ϱ and S being free abelian), there is no extension problem to solve and we conclude that $H_*(\mathcal{W}^I_\varrho, \mathbb{Z})$ is isomorphic to the total complex of the E^2-page, which by (5.5) is simply $H_*(S, \mathbb{Z}) \otimes H_*(\Omega^I_\varrho, \mathbb{Z})$. ☐

6 Complex geometry

In this section, we will study some fundamental features of the complex geometry of the manifolds $\mathcal{W}^I_\varrho$ arising from quotients of domains in flag varieties by images of Anosov representations. As mentioned in the introduction, it is natural to work in a slightly more general setting.

Recall that if $N = G/H$ is a complex homogeneous space of G, then we say a complex manifold $\mathcal{W}$ is an embedded (G, N)-manifold with data (Ω, Γ) and limit set $\Lambda := N - \Omega$ if $\Gamma < G$ acts freely, properly discontinuously, and cocompactly on $\Omega \subset N$ and there is a biholomorphism $\mathcal{W} \simeq \Gamma\backslash\Omega$. For example, if ϱ is P_A-Anosov and I is a balanced ideal of type (P_A, P_D), then the manifold $\mathcal{W}^I_\varrho$ is an embedded $(G, G/P_D)$-manifold with data $(\Omega^I_\varrho, \varrho(\pi))$ and limit set Λ^I_ϱ.

6.1 Non-existence of Kähler metrics and maps to Riemann surfaces

Let m_α denote the α-dimensional Hausdorff measure on N associated to any Riemannian metric. As in Section 4 the particular metric will not matter.

The following classical extension theorem in several complex variables is due to Shiffman:

Theorem 6.1 ([Shi68, Lemma 3])**.** *Let Y be a complex manifold of dimension n and $A \subset Y$ a closed set satisfying $m_{2n-2}(A) = 0$. Then any holomorphic function on $Y - A$ extends to a unique holomorphic function on Y.* □

Using this theorem, we now prove Theorem D from the introduction. We recall the statement:

Theorem D. *Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ) and limit set $\Lambda = N - \Omega$. Suppose that N is simply connected and that $m_{2n-2}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} N$. If X is a compact Riemann surface of positive genus, then every holomorphic map $\mathcal{W} \to X$ is constant.*

Proof. Since N is simply connected, the condition $m_{2n-2}(\Lambda) = 0$ implies that Ω is also simply connected (see e.g. [HW41, Chapter 7]) and hence is biholomorphic to the universal cover of $\mathcal{W}$. Suppose $f : \mathcal{W} \to X$ is a holomorphic map and consider the lifted map $\widetilde{f} : \Omega \to \widetilde{X}$.

By the Köebe-Poincaré uniformization theorem, the universal cover $\widetilde{X}$ is biholomorphic to the unit disk or the complex plane, giving a holomorphic map $\widetilde{f} : \Omega \to \mathbb{C}$. By Theorem 6.1 the map $\widetilde{f}$ extends to a holomorphic map $\widetilde{f} : N \to \mathbb{C}$. However, as N is compact and connected, this extension is constant, hence f is constant as well. □

Combining the previous theorem with criteria for the existence of nonconstant maps from a compact Kähler manifolds to Riemann surfaces, we obtain the obstruction to the existence of Kähler metrics which was stated in the introduction:

Theorem E. *Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ) and limit set Λ. Suppose that N is simply connected and that $m_{2n-2}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} N$. If there exists a surjective homomorphism $\pi_1\mathcal{W} \to \pi_1 S$ for some closed, orientable surface S of genus $g > 1$, then the complex manifold $\mathcal{W}$ does not admit a Kähler metric. In particular, it is not a complex projective variety.*

Proof. Suppose $\mathcal{W}$ admits a Kähler metric. By a theorem due independently to Beauville [Cat91, Appendix] and Siu [Siu87, Theorem 4.7], the hypothesis that there exists a surjective homomorphism $\Gamma \to \pi_1 S$ implies that $\mathcal{W}$ admits a nonconstant holomorphic map to a Riemann surface X of genus greater than one, contradicting Theorem D. □

Applying these theorems to the study of manifolds which are quotients by G-quasi-Fuchsian groups and using the Hausdorff dimension bounds of Section 4, we now give the proof of:

Theorem F. *Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation, where G is a complex simple adjoint Lie group that is not isomorphic to $\mathrm{PSL}_2\mathbb{C}$, and let $P < G$ be a parabolic subgroup. Let $I \subset W$ a balanced and right-W_P-invariant ideal in the Weyl group. Then the associated compact quotient manifold $\mathcal{W}^I_\varrho$ has the following properties:*

(i) *There is no nonconstant holomorphic map from $\mathcal{W}^I_\varrho$ to a compact Riemann surface of positive genus. In particular, $\mathcal{W}$ is not a holomorphic fiber bundle over a Riemann surface.*

(ii) *The complex manifold $\mathcal{W}^I_\varrho$ does not admit a Kähler metric, and in particular it is not a complex projective variety.*

Note that for consistency of notation with the introduction, we are now considering the parabolic pair $(P_A, P_D) = (B, P)$.

Proof. By Theorem 4.9, for such ϱ and I the limit set satisfies $m_{2n-2}(\Lambda^I_\varrho) = 0$. The flag variety G/P is simply connected and thus by the Hausdorff dimension bound so is Ω^I_ϱ, therefore $\pi_1 \mathcal{W}^I_\varrho \simeq \pi_1 S$. Thus statement (i) follows from Theorem D, and (ii) from Theorem E. □

6.2 Picard group

The following theorem of Harvey is an analogue of Shiffman's extension theorem (Theorem 6.1) for holomorphic line bundles and their cohomology:

Theorem 6.2 ([Har74, Theorems 1 and 4])**.** *Let Y be a complex manifold of dimension n and $A \subset Y$ a closed subset satisfying $m_{2n-4}(A) = 0$. Then, every holomorphic line bundle $L \to (Y - A)$ extends uniquely to a holomorphic line bundle on Y.*

Furthermore, if $m_{2n-2k-2}(A) = 0$, then the inclusion map $(Y - A) \hookrightarrow Y$ induces an isomorphism

$$H^i(Y, L) \to H^i(Y - A, L)$$

for all $0 \leqslant i \leqslant k$.

Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ). A line bundle $\mathcal{L}$ on N is Γ-*equivariant* if it carries an action of Γ by bundle automorphisms lifting the action of Γ on N.

Let $p : \Omega \to \Omega/\Gamma \simeq \mathcal{W}$ be the covering map. Given a Γ-equivariant line bundle $\mathcal{L}$ on N, there is a naturally associated line bundle $p^\Gamma_* \mathcal{L}$ on $\mathcal{W}$ which, as a sheaf, is defined by setting

$p_*^\Gamma \mathcal{L}(U)$ to be the space of Γ-invariant sections of $\mathcal{L}|_{p^{-1}(U)}$. This prescription defines the *invariant direct image* homomorphism

$$p_*^\Gamma : \mathrm{Pic}^\Gamma(N) \to \mathrm{Pic}(\mathcal{W}) \tag{6.1}$$

where $\mathrm{Pic}(\mathcal{W})$ is the Picard group of isomorphisms classes of holomorphic line bundles on $\mathcal{W}$, and where $\mathrm{Pic}^\Gamma(N)$ is the group of isomorphism classes of Γ-equivariant line bundles on N.

Using Theorem 6.2 we obtain a sufficient condition for the homomorphism (6.1) to admit a section:

Proposition 6.3. *Let $\mathcal{W}$ be an embedded (G, N)-manifold with data (Ω, Γ) and limit set Λ. Suppose that $m_{2n-4}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} N$. Then for any holomorphic line bundle L on $\mathcal{W}$, we have:*

(i) *The pullback of L to Ω extends uniquely to a Γ-equivariant line bundle on N.*

(ii) *If the pullback of L to Ω is holomorphically trivial and N is simply connected, then L admits a flat, holomorphic connection classified by its holonomy $\chi : \Gamma \to \mathbb{C}^*$.*

Proof. As before let $p : \Omega \to \mathcal{W}$ denote the quotient by Γ. Under the given hypotheses, Theorem 6.2 shows that p^*L extends uniquely to a holomorphic line bundle $\mathcal{L}$ on N. By the uniqueness of the extension, $\mathcal{L}$ is Γ-equivariant, and (i) follows.

Suppose p^*L is holomorphically trivial. Then in a trivialization of p^*L, the action of $\gamma \in \Gamma$ is described by a holomorphic function $q_\gamma : \Omega \to \mathbb{C}^*$. By Shiffman's extension theorem (Theorem 6.1) q_γ extends holomorphically to N, and is therefore constant. Thus the map $\chi : \Gamma \to \mathbb{C}^*$, $\chi(\gamma) = q_\gamma$ is a homomorphism such that $L \simeq (\Omega \times_\chi \mathbb{C})$, and (ii) follows. □

If G is assumed to be semisimple and $P < G$ is a parabolic subgroup, then every line bundle on G/P is G-equivariant. Moreover, the first Chern class gives an isomorphism $c_1 : \mathrm{Pic}(G/P) \xrightarrow{\sim} H^2(G/P, \mathbb{Z})$.

Using the previous theorem, we can now establish the classification of holomorphic line bundles on embedded $(G, G/P)$-manifolds with sufficiently "small" limit sets which was given in the introduction; we recall the statement:

Theorem G. *Let G be a semisimple complex Lie group, $P < G$ a parabolic subgroup, and $\mathcal{W}$ an embedded $(G, G/P)$-manifold with data (Ω, Γ) and limit set Λ. Suppose that $m_{2n-4}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} G/P$. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Pic}(\mathcal{W}) \to \mathrm{Pic}(G/P) \to 0 \tag{1.1}$$

which is canonically split by the invariant direct image homomorphism

$$p_*^\Gamma : \mathrm{Pic}(G/P) \to \mathrm{Pic}(\mathcal{W}). \tag{1.2}$$

Proof. Let $p : \Omega \to \mathcal{W}$ be the covering projection and let L be a holomorphic line bundle on $\mathcal{W}$. By Proposition 6.3(i), the pullback p^*L extends to a unique holomorphic line bundle $\mathcal{L}$ on G/P. Since $p^* \circ p_*^\Gamma(\mathcal{L}) = \mathcal{L}$ we obtain that pullback yields a surjective homomorphism

$$p^* : \mathrm{Pic}(\mathcal{W}) \to \mathrm{Pic}(G/P) \tag{6.2}$$

which is split by the invariant direct image homomorphism.

Next, given $\chi \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, we obtain a homomorphism $\varphi : \mathrm{Hom}(\Gamma, \mathbb{C}^*) \to \mathrm{Pic}(\mathcal{W})$ defined by

$$\varphi(\chi) := \Omega \times_\chi \mathbb{C}.$$

If $\varphi(\chi)$ is holomorphically trivial, then it admits a nowhere vanishing section which corresponds to a χ-equivariant holomorphic map $\Omega \to \mathbb{C}^*$. By Theorem 6.1, this map extends to a χ-equivariant holomorphic function $G/P \to \mathbb{C}^*$ which is constant since G/P is compact and connected. This implies that $\chi(\gamma) = 1$ for all $\gamma \in \Gamma$ which implies that φ is injective.

The fact that $\mathrm{Im}(\varphi) = \mathrm{Ker}(p^*)$ is immediate which verifies the exactness and the proof is complete. □

We remark that the short exact sequence of the previous Theorem *is* the short exact sequence

$$0 \to \mathrm{Pic}_0(\mathcal{W}) \to \mathrm{Pic}(\mathcal{W}) \to \mathrm{NS}(\mathcal{W}) \to 0$$

where $\mathrm{Pic}_0(\mathcal{W})$ is degree zero holomorphic line bundles and $\mathrm{NS}(\mathcal{W})$ is the Neron-Severi group. Therefore, we have shown that there is a natural isomorphism $\mathrm{NS}(\mathcal{W}) \simeq \mathrm{Pic}(G/P)$.

As in the previous subsection, having established this general result for embedded $(G, G/P)$ manifolds with a bound on the dimension of the limit set, Theorem 4.9 shows that the conclusions apply in particular to the manifolds $\mathcal{W}_\varrho^I$ associated to G-quasi-Fuchsian groups if a few exceptional G are excluded. Specifically, we find that $m_{2n-4}(\Lambda_\varrho^I) = 0$ holds as long as G is not of type A_1, A_2, A_3, or B_2. This gives Theorem H.

6.3 Cohomology of holomorphic line bundles

Next we consider the calculation of cohomology of line bundles on embedded $(G, G/P)$-manifolds where G is a complex semisimple Lie group. We will restrict to simply connected G and to $P = B$ to simplify the discussion.

Our results are based on reducing these calculations to the Borel-Bott-Weil theorem, whose statement we recall before proceeding. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a system of simple roots $\Delta \subset \mathfrak{h}^*$; let $L \subset \mathfrak{h}^*$ denote the weight lattice of G and $\varrho \in \mathfrak{h}^*$ half the sum of the positive roots. Associated to each $\lambda \in L$ is a holomorphic line bundle $\mathcal{L}_\lambda = (G \times_\lambda \mathbb{C})$ which is defined by integrating the weight λ to a character $B \to \mathbb{C}^*$. Define $\mathcal{L}^\lambda = \mathcal{L}_{\varrho-\lambda}$. A weight $\lambda \in L$ is *dominant* if $\lambda(\alpha) \geqslant 0$ for all $\alpha \in \Delta$, *strictly dominant* if $\lambda(\alpha) > 0$ for all $\alpha \in \Delta$, and *regular* if its W-orbit contains a strictly dominant weight.

The Borel-Bott-Weil theorem is the following:

Theorem 6.4 ([Bot57]). *The map $\lambda \to \mathcal{L}_\lambda$ is an isomorphism of abelian groups $L \simeq \mathrm{Pic}(G/B)$. Furthermore, the cohomology of $\mathcal{L}^\lambda$ satisfies:*

(i) *If λ is not regular, then $H^i(G/B, \mathcal{L}^\lambda) = 0$ for all $i \geqslant 0$.*

(ii) *If λ is regular, let $w \in W$ be the unique element such that $w(\lambda)$ is dominant. Then $H^i(G/B, \mathcal{L}^\lambda) = 0$ for all $i \neq \ell(w)$, while $H^{\ell(w)}(G/B, \mathcal{L}^\lambda) \neq 0$ and as a G-module this cohomology space is dual to the irreducible representation of G with highest weight $w(\lambda) - \varrho$.* □

Expositions of this Theorem and associated background material can be found in [BE89], [Jan03] (focusing on algebraic groups), or [Sep07] (focusing on compact groups).

Returning to our discussion of an embedded $(G, G/B)$ manifold $\mathcal{W}$, we can cast the problem of determining cohomology of a line bundle on $\mathcal{W}$ in the more general framework of relating the cohomology of a locally free sheaf $\mathcal{F}$ on Y and that of the pullback $p^*\mathcal{F}$ to the universal cover $\widetilde{Y}$. Here the Grothendieck spectral sequence ([Gro57]) can be applied to the composition of the Γ-invariants and global sections functors, giving a cohomology spectral sequence with E_2-page

$$E_2^{p,q} = H^p(\Gamma, H^q(\widetilde{Y}, p^*\mathcal{F})) \tag{6.3}$$

and which converges to the cohomology of $\mathcal{F}$. Using this spectral sequence, we show:

Theorem 6.5. *Let G be a simply connected semisimple complex Lie group, $B < G$ a Borel subgroup, and $\mathcal{W}$ an embedded $(G, G/B)$-manifold with data (Ω, Γ) and limit set Λ. Suppose that $m_{2n-2k-2}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} G/B$ and $k \geqslant 1$.*

Let $\lambda \in L$ be a weight of G and let $p_*^\Gamma : \mathrm{Pic}(G/B) \to \mathrm{Pic}(\mathcal{W})$ denote the invariant direct image functor.

(i) *If λ is not regular, then $H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) = 0$ for all $0 \leqslant i < k$.*

(ii) *If λ is regular and $w(\lambda)$ is dominant for $w \in W$ with $\ell(w) > k$, then $H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) = 0$ for all $0 \leqslant i < k$.*

(iii) *If λ is regular and $w(\lambda)$ is dominant for $w \in W$ with $\ell(w) < k$, then*

$$H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) \simeq \begin{cases} 0 & 0 \leqslant i < \ell(w) \\ H^{i-\ell(w)}\left(\Gamma, H^{\ell(w)}(G/B, \mathcal{L}^\lambda)\right) & \ell(w) \leqslant i < k \end{cases}$$

In particular, the group

$$H^{\ell(w)}(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) \simeq H^0\left(\Gamma, H^{\ell(w)}(G/B, \mathcal{L}^\lambda)\right)$$

is equal to the space of Γ-invariants in the dual of the irreducible G-representation with highest weight $w(\lambda) - \varrho$.

(iv) *In particular, if λ is a regular, dominant weight then*

$$H^i(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) \simeq H^i\left(\Gamma, H^0(G/B, \mathcal{L}^\lambda)\right)$$

for all $0 \leqslant i < k$.

Note that statement (iv) of this theorem is exactly Theorem I from the introduction, since effective line bundles on G/B are exactly those of the form $\mathcal{L}^\lambda$ for regular, dominant λ.

Proof. By Harvey's extension theorem (Theorem 6.2), the hypothesis on Hausdorff dimension gives an isomorphism

$$H^i(G/B, \mathcal{L}^\lambda) \simeq H^i(\Omega, \mathcal{L}^\lambda)$$

for all $0 \leqslant i \leqslant k$. Since $k \geqslant 1$, the same hypothesis ensures that Ω is simply connected, and thus is the universal cover of $\mathcal{W}$. Thus the spectral sequence (6.3) applies and its E_2-page is determined up to the k-th row:

$$\begin{array}{c|cccc}
k & H^0(\Gamma, H^k(G/B, \mathcal{L}^\lambda)) & H^1(\Gamma, H^k(G/B, \mathcal{L}^\lambda)) & \cdots & H^{\mathrm{cd}(\Gamma)}(\Gamma, H^k(G/B, \mathcal{L}^\lambda)) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
1 & H^0(\Gamma, H^1(G/B, \mathcal{L}^\lambda)) & H^1(\Gamma, H^1(G/B, \mathcal{L}^\lambda)) & \cdots & H^{\mathrm{cd}(\Gamma)}(\Gamma, H^1(G/B, \mathcal{L}^\lambda)) \\
0 & H^0(\Gamma, H^0(G/B, \mathcal{L}^\lambda)) & H^1(\Gamma, H^0(G/B, \mathcal{L}^\lambda)) & \cdots & H^{\mathrm{cd}(\Gamma)}(\Gamma, H^0(G/B, \mathcal{L}^\lambda)) \\
\hline
 & 0 & 1 & \cdots & \mathrm{cd}(\Gamma)
\end{array}$$

Here $\mathrm{cd}(\Gamma) \in \mathbb{Z}^{\geqslant 0}$ denotes the cohomological dimension of Γ; by definition of this integer, entries in the E_2 page to the right of those indicated here are zero. Meanwhile, entries above the k-th row involve groups of the form $H^j(\Omega, \mathcal{L}^\lambda)$ we do not know how to compute.

The entire proposition now follows simply by applying the Borel-Bott-Weil theorem. For instance, if λ is not regular, then all the coefficients appearing in the above rectangle of the E_2-page vanish, which immediately yields statement (i). The same is true if λ is regular, but the $w \in W$ such that $w(\lambda)$ is dominant satisfies $\ell(w) > k$, from which statement (ii) follows.

In the case that $\ell(w) < k$, only the $\ell(w)$-th row is nonzero, so all relevant differentials are zero. Using the description of the entries in this row from the Borel-Bott-Weil theorem, statements (iii) and (iv) follow. This completes the proof. $\square$

We now explain a connection between these computations and classical questions in geometric invariant theory (a theme which is also explored in [KLP13] and [ST15]). Note that the semisimple group G is an affine algebraic group over $\mathbb{C}$. The representation ν of G on $H^0(G/B, \mathcal{L})$ is a rational representation. Therefore, given a subspace $V \subset H^0(G/B, \mathcal{L})$, its stabilizer

$$\{g \in G \,:\, \nu(g)s - s = 0 \text{ for all } s \in V\}$$

is Zariski closed. We record this in the following proposition.

Proposition 6.6. *Let G be a simply connected, complex semisimple Lie group and $\lambda \in L$ a regular dominant weight. If $\Gamma < G$ is a subgroup with Zariski closure $Q < G$, then*

$$H^0\left(\Gamma, H^0(G/B, \mathcal{L}^\lambda)\right) = H^0\left(Q, H^0(G/B, \mathcal{L}^\lambda)\right),$$

where the right hand side is the space of Q-invariant sections of $\mathcal{L}^\lambda$. $\square$

This leads to the following result.

Theorem 6.7. *Let G be a simply connected semisimple complex Lie group, $B < G$ a Borel subgroup, and $\mathcal{W}$ an embedded $(G, G/B)$-manifold with data (Ω, Γ) and limit set Λ. Let $Q < G$ denote the Zariski closure of Γ. Suppose that $m_{2n-4}(\Lambda) = 0$ where $n = \dim_{\mathbb{C}} G/B$.*

Let $\lambda \in L$ be a regular dominant weight and let $p_*^\Gamma : \mathrm{Pic}(G/B) \to \mathrm{Pic}(\mathcal{W})$ denote the invariant direct image homomorphism. Then:

$$H^0(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) \simeq H^0\left(Q, H^0(G/B, \mathcal{L}^\lambda)\right),$$

where the latter is the space of Q-invariant sections. In particular, if Γ is Zariski dense in G, then $H^0(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) = 0$.

Proof. The isomorphisms

$$H^0(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda)) \simeq H^0\left(\Gamma, H^0(G/B, \mathcal{L}^\lambda)\right) = H^0\left(Q, H^0(G/B, \mathcal{L}^\lambda)\right)$$

follow from Theorem 6.5 and Proposition 6.6, respectively. If $Q = G$, then the irreducibility $H^0(G/B, \mathcal{L}^\lambda)$ as a G-representation implies that the space of G-invariants is trivial. $\square$

In the ensuing applications, we will give explicit examples where $H^0(\mathcal{W}, p_*^\Gamma(\mathcal{L}^\lambda))$ is non-vanishing.

6.4 Applications

We will now present some applications of the previous calculations: in particular we show that, excluding some low dimensional cases, every manifold arising from a G-quasi-Fuchsian representation admits a meromorphic function. In this section, we will return to the notation $\mathcal{L}_\lambda = G \times_\lambda \mathbb{C}$ and note that $\mathcal{L}_{k\lambda} = \mathcal{L}_\lambda^k$ where the latter is the k-th tensor power. Given a subgroup $H < G$, we say that $\mathcal{L}_\lambda$ is *twice H-ample* if some power $\mathcal{L}_{k\lambda}$ admits a pair of nonproportional H-invariant sections.

We begin with the following result which follows quickly from results in [ST15].

Theorem 6.8. *Let G be an adjoint complex simple Lie group not of type A_1, A_2, or B_2 with principal three-dimensional embedding $\iota_G : \mathrm{PSL}_2\mathbb{C} \to G$. Let $\mathfrak{S} = \iota_G(\mathrm{PSL}_2\mathbb{C})$. Then every ample line bundle $\mathcal{L}$ on G/B is twice $\mathfrak{S}$-ample.*

Proof. First, recall that ample line bundles on G/B are of the form $\mathcal{L}_{-\lambda}$ for $\lambda \in L$ some regular, dominant weight. Consider the graded ring $R(\lambda) = \bigoplus_{k>0} H^0(G/B, \mathcal{L}_{-k\lambda})$ and the subring $R(\lambda)^{\mathfrak{S}}$ of $\mathfrak{S}$-invariant elements. Define the subset $Y(\lambda) \subset G/B$ by

$$Y(\lambda) := \{x \in G/B \, : \, s(x) = 0 \text{ for every } s \in R(\lambda)^{\mathfrak{S}}\}.$$

Under the hypotheses, it is shown in [ST15] that the complex codimension of $Y(\lambda)$ is at least two. Since the vanishing locus of a non-zero holomorphic section has complex codimension one, this implies that there exists a pair of $\mathfrak{S}$-invariant sections $s_i \in H^0(G/B, \mathcal{L}_{-k_i\lambda})$ for $i = 1, 2$ with distinct vanishing loci. Then $s_1^{k_2}$ and $s_2^{k_1}$ are nonproportional sections of $\mathcal{L}_{-(k_1+k_2)\lambda}$. ☐

This leads to a proof of the following theorem stated in the introduction:

Theorem J. *Suppose G is an adjoint complex simple Lie group not of type A_1, A_2, A_3 or B_2. Let $\varrho : \pi_1 S \to G$ be a G-quasi-Fuchsian representation with image Γ, and let I be a balanced ideal in the Weyl group W of G. Let $\mathcal{W}_\varrho^I$ denote the embedded $(G, G/B)$-manifold associated to these data. For any ample line bundle $\mathcal{L}$ on G/B, the following properties hold:*

(i) *There exists a $k > 0$ such that*

$$H^0(\mathcal{W}_\varrho^I, p_*^\Gamma(\mathcal{L}^k)) \simeq H^0(\Gamma, H^0(G/B, \mathcal{L}^k)) \neq 0.$$

(ii) *The manifold $\mathcal{W}_\varrho^I$ admits a non-constant meromorphic function.*

Proof. By Theorem 4.9, the exclusion of types A_1, A_2, A_3 and B_2 implies that $m_{2n-4}(\Lambda_\varrho^I) = 0$. Recall that ample line bundles on G/B are of the form $\mathcal{L}_{-\lambda}$ for $\lambda \in L$ a regular dominant weight. By Theorem 6.5(iv),

$$H^0(\mathcal{W}_\varrho^I, p_*^\Gamma(\mathcal{L}_{-k\lambda})) \simeq H^0(\Gamma, H^0(G/B, \mathcal{L}_{-k\lambda}))$$

which is nonvanishing for some $k > 0$ by Theorem 6.8. Here, we have used that every $\mathfrak{S}$-invariant section is Γ-invariant (since $\Gamma \subset \mathfrak{S}$). Thus statement (i) follows.

By Theorem 6.8 the ample bundle $\mathcal{L}_{-\lambda}$ is twice Γ-ample, thus there exists $k > 0$ such that $\mathcal{L}_{-k\lambda}$ has a pair of Γ-invariant non-proportional holomorphic sections. The quotient of these sections is a non-constant Γ-invariant meromorphic function on G/B, hence its restriction to Ω^I_ϱ descends to a non-constant meromorphic function on $\mathcal{W}^I_\varrho$, giving (ii). $\square$

As a final application of our sheaf cohomology calculations, we consider the Kodaira dimension of embedded $(G, G/B)$-manifolds. Recall that a compact complex manifold Y with canonical bundle K_Y is said to have *Kodaira dimension* $-\infty$, denoted $\kappa(Y) = -\infty$, if $H^0(Y, K_Y^d)$ vanishes for all $d > 0$. Because the flag variety G/B is rational, it has $\kappa(G/B) = -\infty$. The same holds for embedded $(G, G/B)$-manifolds with sufficiently small limit sets:

Theorem 6.9. *Let G be a simply connected semisimple complex Lie group of rank at least two. Let $B < G$ a Borel subgroup, and $\mathcal{W}$ an embedded $(G, G/B)$-manifold with data (Ω, Γ) and limit set Λ. Suppose $m_{2n-4}(\Lambda) = 0$ where n is the complex dimension of G/B. Then $\kappa(\mathcal{W}) = -\infty$.*

Proof. The canonical line bundle of G/B is isomorphic to $\mathcal{L}^{-\varrho}$ where ϱ is half the sum of the positive roots. Therefore, we have

$$K^d_{\mathcal{W}} \simeq p^\Gamma_*(\mathcal{L}^{(1-2d)\varrho}).$$

For any $d > 0$, the weight $(1 - 2d)\varrho$ is regular and $w_0((1 - 2d)\varrho)$ is dominant, where w_0 is the longest element of the Weyl group. Therefore, by Theorem 6.5(ii) we have

$$H^0(\mathcal{W}, K^d_{\mathcal{W}}) \simeq H^0(\mathcal{W}, p^\Gamma_*(\mathcal{L}^{(1-2d)\varrho})) = 0$$

for all $d > 0$ provided that $\ell(w_0) > 1$, which is the case since the rank of G is at least two. $\square$

Note that the corresponding statement fails for $G \simeq \mathrm{SL}_2\mathbb{C}$ since Riemann surfaces of higher genus can be obtained as embedded $(G, G/B) = (\mathrm{SL}_2\mathbb{C}, \mathbb{P}^1_{\mathbb{C}})$ manifolds, and the canonical bundle of such a Riemann surface has nontrivial sections.

7 Examples and complements

In this final section we return to the topological considerations of Section 5 and discuss some specific examples of balanced ideals, domains, and quotient manifolds for various complex simple Lie groups G and parabolic pairs (P_A, P_D). (The survey [KL17] also gives examples of balanced ideals, including some that belong to the infinite families constructed below.)

7.1 The lower half of W

We begin by indicating how certain ideals can be constructed easily from the length function on the Weyl group W. Since $x < y$ implies $\ell(x) < \ell(y)$, the set

$$W_{\leqslant L} := \{x \,:\, \ell(x) \leqslant L\}$$

is an ideal in W for any integer L, and this ideal is minimally generated by $\ell^{-1}(L)$. Generalizing this, if J is a subset of $\ell^{-1}(L+1)$, then $W_{\leqslant L} \cup J$ is also an ideal, and the minimal generating set of this ideal contains J.

This construction can always be used to produce a balanced ideal. Define the *lower half* of W to be the ideal

$$I_{\frac{1}{2}} = W_{\leqslant \frac{1}{2}\ell(w_0)}.$$

Since $\ell(w_0 x) = \ell(w_0) - \ell(x)$, it is immediate that this ideal is balanced if $\ell(w_0) = \dim_{\mathbb{C}} G/B$ is odd, which is the case for all simple G of type $B_n = \mathrm{PO}_{2n+1}\mathbb{C}$, $C_n = \mathrm{PSp}_{2n}\mathbb{C}$, or E_7, and for type $A_n = \mathrm{PSL}_{n+1}\mathbb{C}$ when n is 1 or 2 mod 4.

In such cases, considering $I_{\frac{1}{2}}$ as an ideal of type (B, B), it gives a model thickening $\Phi_{\frac{1}{2}} := \Phi^{I_{\frac{1}{2}}} \subset G/B$ and domain of discontinuity $\Omega_{\frac{1}{2}} \subset G/B$ for B-Anosov representations. Suppose $\ell(w_0) = 2k+1$ for $k \in \mathbb{Z}$. Then the model thickening has the same Betti numbers as G/B itself in the range $1 \ldots 2k$, i.e.

$$r_i = b_i(\Phi_{\frac{1}{2}}) = b_i(G/B) = |\ell^{-1}(i)| \text{ for } i \leqslant 2k.$$

Applying Corollary 5.7 gives a particularly simple expression for the Betti numbers of the domain of discontinuity:

$$b_i(\Omega_{\frac{1}{2}}) = \begin{cases} b_i(G/B) & \text{if } i < 2k \\ 2b_{2k}(G/B) & \text{if } i = 2k \\ b_{4k-i}(G/B) & \text{if } i > 2k \end{cases}$$

By Theorem 5.9 there is a corresponding formula for the homology of the compact quotient manifolds.

If $\ell(w_0) = 2k$ is even, the construction can be modified to produce a balanced ideal. Note that the "middle" length $W_{\mathrm{mid}} := \ell^{-1}(k)$ is mapped to itself under left multiplication by w_0. Let $J \subset W_{\mathrm{mid}}$ be a subset containing one element of each w_0-orbit. Then the set

$$I_{\frac{1}{2},J} = W_{\leqslant (k-1)} \cup J$$

is a balanced ideal whose minimal generating set contains J. (In some examples, $I_{\frac{1}{2},J}$ is in fact generated by J, while in other cases there are additional generators of length $k-1$.)

Since there are $2^{|W_{\mathrm{mid}}|/2}$ such sets J, this gives a large collection of balanced ideals, all of which have the same number of elements of each length. The corresponding generalizations of the Betti number formulas given above are

$$r_i = b_i(\Phi_{\frac{1}{2},J}) = \begin{cases} b_i(G/B) & i < 2k \\ \frac{1}{2}b_i(G/B) & i = 2k \\ 0 & \text{else} \end{cases}$$

and by Corollary 5.7,

$$b_i(\Omega_{\frac{1}{2},J}) = \begin{cases} b_i(G/B) & \text{if } i < 2k-2 \\ b_{2k-2}(G/B) + \frac{1}{2}b_{2k}(G/B) & \text{if } i \in \{2k-2, 2k\} \\ b_{4k-2-i}(G/B) & \text{if } i > 2k. \end{cases} \tag{7.1}$$

7.2 Constructions for $\mathrm{PSL}_n\mathbb{C}$

In preparation for the next two types of examples, we recall how some of the combinatorial and Lie-theoretic notions specialize to the case $G = A_{n-1} = \mathrm{PSL}_n\mathbb{C}$; general references for this material include [BB05] (concerning Weyl groups), [LG01] [Bri05] (concerning flag varieties), and [Ful97] (concerning both).

We choose the Borel $B < G = \mathrm{PSL}_n\mathbb{C}$ consisting of the upper-triangular matrices. The manifold G/B is G-equivariantly identified with the set of complete flags $F = (F_1, \ldots F_{n-1})$, i.e. $F_1 \subset \ldots \subset F_{n-1} \subset \mathbb{C}^n$ and $\dim_{\mathbb{C}} F_k = k$. We denote by E the standard flag of $\mathbb{C}^n$ in which $E_k = \mathrm{span}\{e_1, \ldots, e_k\}$, which corresponds to $eB \in G/B$; here $e_1, \ldots, e_n$ is the standard ordered basis of $\mathbb{R}^n$.

Standard parabolic subgroups $P < G$ are stabilizers of partial flags within E, with associated quotients G/P parameterizing all flags of that type. An example we will focus on is $P_{1,n-1}$, the *incidence parabolic*, which is defined as the stabilizer of (E_1, E_{n-1}). Thus $G/P_{1,n-1}$ is the set of pairs (ℓ, H) of a line and a containing hyperplane.

The Weyl group $W = W(\mathrm{PSL}_n\mathbb{C})$ is isomorphic to the symmetric group S_n, with the roots (respectively, simple roots) of G corresponding to transpositions (respectively, transpositions of adjacent elements). We identify a permutation $x \in S_n$ with the tuple $(x(1), x(2), \ldots, x(n))$. Thus the longest element is $w_0 = (n, n-1, \ldots, 1)$.

The Weyl group $W_{1,n-1}$ of $P_{1,n-1}$ consists of permutations $w \in S_n$ with $w(1) = 1$ and $w(n) = n$. Thus, the cosets space $W/W_{1,n-1}$ consists of classes of permutations $W(i,j) = \{(i, *, \ldots, *, j)\} \subset S_n$ for $i \neq j$.

The Chevalley-Bruhat order has a simple description in terms of permutations. For $w \in S_n$ we define the set of *ascents* of w to be

$$A(w) := \{i \,:\, w(i) < w(i+1)\}.$$

This is a subset of $\{1, 2, \ldots, n-1\}$. We also denote by $w_{i,j}$ the j-th smallest element of the set $\{w(1), \ldots, w(i)\}$. Then:

Theorem 7.1 ([BB05, Theorem 2.6.3(iii)]). *Elements $x, y \in S_n$ satisfy $x \leqslant y$ if and only if $x_{i,j} \leqslant y_{i,j}$ for all $i \in A(y)$ and all $j \leqslant i$.* $\square$

Note that this characterizes elements of the ideal $\langle y \rangle = \{x : x \leqslant y\}$ by an explicit set of inequalities. There is a corresponding formula for the length of an element $w \in S_n$ as its number of *inversions* (see [BB05, Proposition 1.5.2]):

$$\ell(x) = |\{(i,j) : i < j \text{ and } \sigma(i) > \sigma(j)\}| . \tag{7.2}$$

The Schubert variety $X_w = \overline{BwB} \subset G/B$ is defined by an explicit set of dimension inequalities depending on the permutation w; precisely, we have:

Theorem 7.2 ([Ful97, Section 10.5]). *The Schubert variety X_w consists of the flags $(F_1, \ldots, F_n)$ such that*

$$\dim(F_p \cap E_q) \geqslant |\{(i,j) : i \leqslant p,\ w(j) \leqslant q\}| . \qquad \square$$

Finally, we note that the partial flag variety $G/P_{1,n-1} = \{(\ell, H)\}$ can be embedded as a hypersurface in $\mathbb{P}^{n-1}_{\mathbb{C}} \times (\mathbb{P}^{n-1}_{\mathbb{C}})^*$, which we call the *incidence variety*, consisting of pairs of a vector $x \in \mathbb{C}^n$ and a linear form $\xi \in (\mathbb{C}^n)^*$ such that $\xi(x) = 0$, modulo the action of $\mathbb{C}^* \times \mathbb{C}^*$. Here (x, ξ) corresponds to the flag $(\mathbb{C} \cdot x, \ker \xi)$. Using the Theorem above, one can check that in this realization the Schubert variety $X_{W(i,j)} \subset G/P_{1,n-1}$ is cut out by the equations $x_{i+1} = \ldots = x_n = \xi_1 = \ldots = \xi_{j-1} = 0$.

7.3 The $(1, n-1)$-examples

In this section we describe how certain domains studied by Guichard-Wienhard in [GW12, Section 10.2.2] are represented in the Kapovich-Leeb-Porti formalism (i.e. by Chevalley-Bruhat ideals), and what is obtained by applying the results of Section 5 to these examples.

We define the *incidence ideal* to be the subset of S_n given by

$$I_{1,n-1} = \{x \in S_n : x(1) < x(n)\}.$$

Equivalently, this is a union of $W_{1,n-1}$ cosets, $I_{1,n-1} = \bigcup_{i<j} W(i,j)$.

For $1 \leqslant k \leqslant n-1$, let $z_k \in S_n$ be defined by

$$z_k(i) = \begin{cases} k & \text{if } i = 1 \\ k+1 & \text{if } i = n \\ n-i+2 & \text{if } 1 < i \leqslant n-k \\ n-i & \text{otherwise.} \end{cases}$$

Equivalently (and perhaps more transparently) z_k is defined by the unique tuple $(k, \ldots, k+1)$ in which the omitted elements appear in decreasing order. Note that $z_k \in I_{1,n-1}$, and that z_k is the unique longest element in the coset $W(k, k+1)$.

Theorem 7.3. *The set $I_{1,n-1} \subset S_n$ is a balanced and right $W_{1,n-1}$-invariant ideal of the Chevalley-Bruhat order on S_n. It is minimally generated by $\{z_1, z_2, \ldots, z_{n-1}\}$.*

Proof. Since $(w_0x)(i) = n+1-x(i)$ it is immediate that left multiplication by w_0 exchanges $I_{1,n-1}$ with its complement. Thus if this set is an ideal, then it is balanced. We have already seen that $I_{1,n-1}$ is a union of left $W_{1,n-1}$-cosets (and hence right-$W_{1,n-1}$-invariant).

Next, we claim that the Chevalley-Bruhat order satisfies

$$x \leqslant z_k \quad \text{if and only if} \quad x(1) \leqslant k \text{ and } x(n) > k. \tag{7.3}$$

Before proving this, we derive the rest of the statements of the Theorem from it. An element $x \in W$ satisfies the right hand side of (7.3) for some k if and only if $x(1) < x(n)$, hence the condition above is equivalent to the statement that $I_{1,n-1}$ is the union of the principal ideals $\langle z_k \rangle$ for $k = 1, \ldots, n-1$, and in particular is an ideal. It is straightforward to calculate from (7.2) that $\ell(z_k) = \frac{1}{2}(n-1)(n-2)$ for all k, so these elements are pairwise incomparable and of maximal length within $I_{1,n-1}$. This shows $\{z_1, z_2, \ldots, z_{n-1}\}$ is the minimal generating set.

Finally we prove (7.3) using Theorem 7.1. First suppose that $1 < k < n-1$. Then $A(z_k) = \{1, n-1\}$ and we find $x \leqslant z_k$ if and only if

$$x(1) = x_{1,1} \leqslant (z_k)_{1,1} = z_k(1) = k$$

and

$$x_{n-1,j} \leqslant (z_k)_{n-1,j} \;\text{ for }\; j \leqslant n-1.$$

Since $\{x(1), \ldots, x(n-1)\} = \{1, \ldots, n\} - x(n)$ (and similarly for z_k), the second set of inequalities is equivalent to $x(n) \geqslant z_k(n) = k+1$, or equivalently $x(n) > k$, as desired. The cases $k = 1$ and $k = n-1$ are similar, except that z_k then has only one ascent. We omit the straightforward verification that the argument above still applies in these cases. ☐

Using the right-invariance of $I_{1,n-1}$ we can apply the Kapovich-Leeb-Porti construction with $P_A = B$ and $P_D = P_{1,n-1}$ to obtain a limit set $\Lambda_{1,n-1} := \Lambda_\varrho^{I_{1,n-1}}$ and cocompact domain of discontinuity $\Omega_{1,n-1} := \Omega_\varrho^{I_{1,n-1}}$ in the incidence variety $G/P_{1,n-1}$ for a B-Anosov representation $\varrho : \pi \to G$ of a word hyperbolic group π.

Applying Theorem 7.2 to z_k we find that the associated Schubert variety $X_{z_k} \subset G/B$ is characterized by dimension inequalities $\dim(F_1 \cap E_k) \geqslant 1$ and $\dim(E_k \cap F_{n-1}) \geqslant k$. Projecting to $G/P_{1,n-1}$ we obtain the Schubert variety

$$X_{W(k,k+1)} = X_{z_k W_{1,n-1}} = \{(F_1, F_{n-1}) \,:\, F_1 \subset E_k \subset F_{n-1}\}.$$

Taking the union of these sets over k gives the model thickening $\Phi_{1,n-1} := \Phi^{I_{1,n-1}}$ in $G/P_{1,n-1}$, and the limit set itself is given by

$$\Lambda_{1,n-1} = \bigcup_{t \in \partial_\infty \pi} \{(F_1, F_{n-1}) \,:\, \exists k,\; F_1 \subset \xi_k(t) \subset F_{n-1}\},$$

where $\xi_k(t)$ is the k-dimensional component of the flag corresponding to $\xi(t) \in G/B$. This is the domain constructed in [GW12, Section 10.2.2]. Using the results of Section 5 we can now derive a closed formula for the Betti numbers of $\Omega_{1,n-1}$ in the case of a G-Fuchsian representation.

Theorem 7.4. *This domain of discontinuity $\Omega_{1,n-1} \subset G/P_{1,n-1}$ in the incidence variety associated to a G-Fuchsian representation $\varrho : \pi_1 S \to \mathrm{PSL}_n\mathbb{C}$ satisfies*

$$b_{2k}(\Omega_{1,n-1}) = \begin{cases} 2n-2 & \text{if } k = n-2 \\ \max(0, n-1-|n-k-2|) & \text{else.} \end{cases}$$

Hence its Poincaré polynomial is

$$p(x) = \sum_i b_i x^i = \frac{\left(1-t^{2(n-1)}\right)^2}{(1-t^2)^2} + (n-1)t^{2n-4}.$$

Proof. Recall that r_k is the number of elements of $I/W_{1,n-1}$ of length k, and that $I/W_{1,n-1}$ consists of the cosets $W(i,j)$ with $i<j$. By (7.2), the element of $W(i,j)$ of minimal length

$$(i, 1, 2, \ldots, \widehat{i}, \ldots, \widehat{j}, \ldots, n, j) \in W(i,j)$$

has length $n+i-j-1$, hence r_k is the number of pairs (i,j) with $1 \leqslant i < j \leqslant n$ and $n+i-j-1 = k$. Such pairs exist for $0 \leqslant k \leqslant n-2$, and enumerating them we find

$$r_k = \begin{cases} k+1 & \text{if } 0 \leqslant k \leqslant n-2 \\ 0 & \text{else.} \end{cases}$$

Since $\dim_{\mathbb{C}} \mathcal{F}_{1,n-1} = 2n-3$, Corollary 5.7 gives $b_{2k}(\Omega) = r_k + r_{2n-4-k}$. Substituting the formula for r_k we find that for all k except $n-2$, only one of the terms is nonzero. Considering the various cases for k we find

$$b_{2k}(\Omega_{1,n-1}) = \begin{cases} k+1 & \text{if } 0 \leqslant k < n-2 \\ 2n-2 & \text{if } k = n-2 \\ 2n-3-k & \text{if } n-2 < k \leqslant 2n-4 \\ 0 & \text{if } k > 2n-4 \end{cases}$$

which is easily seen to be equivalent to the formula in the Theorem. We omit verification of the corresponding closed form for $p(x)$. □

Having studied $I_{1,n-1}$ in some detail, it is of course natural to consider its inverse $I_{1,n-1}^{-1} = \{w : w^{-1}(1) < w^{-1}(n)\}$ which is balanced and left $W_{1,n-1}$-invariant. This ideal gives a domain of discontinuity in G/B for any $P_{1,n-1}$-Anosov representation (a construction which was also considered in [GW12]).

An important class of examples of $P_{1,n-1}$-Anosov representations which are not B-Anosov arise from holonomies of strictly convex projective manifolds when $n \geqslant 4$. Recall that a representation $\varrho : \pi \to \mathrm{PSL}_n\mathbb{R}$ is the holonomy of a strictly convex projective structure if there exists a strictly convex $\Gamma = \varrho(\pi)$-invariant open set $\Omega \subset \mathbb{P}(\mathbb{R}^n)$ upon which the action of Γ is properly discontinuous and cocompact. The boundary of this domain is C^1, and therefore by taking points along with their supporting hyperplanes, we obtain a limit curve

$$\partial_\infty \pi \to \mathrm{PSL}_n\mathbb{R}/\left(P_{1,n-1} \cap \mathrm{PSL}_n\mathbb{R}\right) \subset G/P_{1,n-1}.$$

Examples for surface groups can be obtained by restricting such a representation to a quasi-convex surface subgroup $\pi_1 S < \pi$.

While the combinatorial considerations applied to $I_{1,n-1}$ can for the most part be adapted to study these domains in G/B arising from $I_{1,n-1}^{-1}$, the resulting Betti number formulas are substantially more complicated, and we omit discussion of them.

7.4 The $2n$ examples: Principal balanced ideals

All of the ideals discussed so far in this section have large minimal generating sets; this follows, for example, from their having many elements of maximal length. In this subsection we describe a family of examples of balanced ideals that are also principal, i.e. generated by a single element. In more geometric terms, these correspond to model thickenings given by a single Schubert variety.

Let $G = \mathrm{PSL}_{2n}\mathbb{C}$, so that $W \simeq S_{2n}$. We have:

Theorem 7.5. *The set $I_{2n} := \{w \in S_{2n} \ : \ w(2n) > n\}$ is a principal, balanced ideal. In fact, $I_{2n} = \langle \lambda \rangle$ where $\lambda = (2n, 2n-1, \dots, \widehat{n+1}, \dots, 2, 1, n+1)$.*

Proof. Since $(w_0 x)(i) = 2n + 1 - x(i)$, it is immediate from the definition that I_{2n} and its complement are exchanged by left multiplication by w_0. Thus if I_{2n} is an ideal, it is balanced, and it suffices to show $I_{2n} = \langle \lambda \rangle$.

Examining the explicit form of λ we see there is a single ascent, $A(\lambda) = \{2n-1\}$. Applying Theorem 7.1 and computing $\lambda_{2n-1,j}$ we find that $x \in \langle \lambda \rangle$ if and only if

$$\begin{aligned} x_{2n-1,j} &\leqslant j && \text{for } j \leqslant n \\ x_{2n-1,j} &\leqslant j+1 && \text{for } j > n \end{aligned} \tag{7.4}$$

But note that $\{x(1), \dots, x(2n-1)\} = \{1, \dots, 2n\} - \{x(2n)\}$, hence for all x we have

$$x_{2n-1,j} = \begin{cases} j & \text{if } j < x(2n) \\ j+1 & \text{if } j \geqslant x(2n) \end{cases}$$

Comparing this to (7.4), we see that $x \in \langle \lambda \rangle$ if and only if $x(2n) < n$, as desired. $\square$

As mentioned above, because I_{2n} is principal, the associated model thickening $\Phi_{2n} := \Phi^{I_{2n}} \subset G/B$ is the Schubert variety X_λ. While Schubert varieties can in general have singularities, this one is smooth: This is immediate from the pattern avoidance criterion of Lakshmibai-Sandhya [LS90], or it can be verified from the description of X_λ using dimension inequalities for flags. The latter will give a more detailed description and allow us to compute the Poincaré polynomial of $\Omega_{2n} := \Omega^{I_{2n}}$:

Theorem 7.6. *The domain of discontinuity Ω_{2n} has Poincaré polynomial*

$$\frac{(1+t^{2n-2})(1-t^{2n})}{(1-t^2)^{2n-1}} \prod_{i=1}^{2n-2} \left(1 - t^{2(i+1)}\right).$$

Proof. For brevity, in this proof we denote by $\mathcal{F}(m)$ the full flag variety of $\mathbb{C}^m$ and by $\mathcal{F}(i_1, \dots, i_k; m)$ the variety of partial flags in $\mathbb{C}^m$ with components of dimensions $i_1 < i_2 < \dots < i_k$. Each such space is a smooth manifold. We write $p[X](t)$ for the Poincaré polynomial of a space X.

The projection $\pi : (F_1, \dots, F_{m-1}) \mapsto (F_1, \dots, F_k)$ is a smooth fibration of $\mathcal{F}(m)$ over $\mathcal{F}(1, \dots, k; m)$ with fiber diffeomorphic to $\mathcal{F}(m-k)$. Furthermore, applying the Serre spectral sequence shows that this bundle is homologically trivial. Thus the Poincaré polynomial of the base of this bundle satisfies

$$p[\mathcal{F}(1, \dots, k; m)] = \frac{p[\mathcal{F}(m)]}{p[\mathcal{F}(m-k)]}. \tag{7.5}$$

Applying Theorem 7.2 to the permutation λ we find

$$\Phi_{2n} = X_\lambda = \{(F_1, \dots, F_{2n-1}) \,:\, F_n \subset E_{2n-1}\}.$$

Considering the fibration $\mathcal{F}(2n) \to \mathcal{F}(1 \dots, n; 2n)$ (i.e. taking $m = 2n$ and $k = n$ above), this description of Φ_{2n} is equivalent to identifying it with the preimage $\pi^{-1}(Y)$ of $Y = \{(F_1, \dots, F_n) \,:\, F_n \subset E_{2n-1}\} \simeq \mathcal{F}(1, \dots, n; 2n-1)$. Thus Φ_{2n} is a smooth fiber bundle over $\mathcal{F}(1, \dots, n; 2n-1)$ with fiber $\mathcal{F}(n)$. Again applying the Serre spectral sequence shows this bundle is homologically trivial and we obtain

$$p[\Phi_{2n}] = p[\mathcal{F}(1, \dots, n; 2n-1)]p[\mathcal{F}(n)]$$

Using (7.5) with $m = 2n-1$ and $k = n$ we find $p[\mathcal{F}(1, \dots, n; 2n-1)] = p[\mathcal{F}(2n-1)]/p[\mathcal{F}(n-1)]$ and thus

$$p[\Phi_{2n}] = \frac{p[\mathcal{F}(2n-1)]p[\mathcal{F}(n)]}{p[\mathcal{F}(n-1)]}.$$

Substituting the classical formula for the Poincaré polynomial of the flag variety itself (see e.g. [Mac72]),

$$p[\mathcal{F}(m)](t) = (1-t^2)^{1-n} \prod_{i=1}^{m-1} (1 - t^{2(i+1)}),$$

and simplifying we obtain

$$p[\Phi_{2n}](t) = \frac{(1-t^{2n})}{(1-t^2)^{2n-1}} \prod_{i=1}^{2n-2} (1-t^{2(i+1)}).$$

It follows from (7.2) that $\ell(\lambda) = \ell(w_0) - n$. Since it is a smooth manifold, the model thickening Φ_{2n} satisfies Poincaré duality in this dimension. In terms of the number r_k of elements of I of length k, this means

$$r_k = r_{L-n-k}$$

where $L = \ell(w_0)$, and the formula of Corollary 5.7 simplifies in this case to

$$b_{2k}(\Omega_{2n}) = r_k + r_{k-(n-1)}.$$

Returning to Poincaré polynomials, this shows

$$p[\Omega_{2n}](t) = (1 + t^{2n-2}) p[\Phi_{2n}](t),$$

and substituting the expression for $p[\Phi_{2n}](t)$ obtained above, the Theorem follows. $\square$

7.5 Homotopy types

For most complex adjoint groups G there are many balanced ideals in $I \subset W$; it is natural to ask whether these correspond to topologically distinct quotient manifolds $\mathcal{W}^I$. We will verify this for two of the Chevalley-Bruhat ideal examples studied thus far, applied to G-Fuchsian representations:

Theorem 7.7. *Let $G = \mathrm{PSL}_{2n}\mathbb{C}$ where $n = 2j+1$, $j \in \mathbb{Z}$. Let $I_{\frac{1}{2}}, I_{2n} \subset W$ denote, respectively, the lower half and principal balanced ideals constructed above. Let $\varrho : \pi_1 S \to G$ be a G-Fuchsian representation. Then the quotient manifolds $\mathcal{W}_{\frac{1}{2}}$ and $\mathcal{W}_{2n}$ associated to ϱ are not homotopy equivalent.*

Proof. In this case $L = \ell(w_0) = 2k+1$ where $k = j(4j+3)$. By Corollary 5.7 we have for any balanced ideal I that

$$b_{2k}(\Omega^I) = 2r_k(I) = 2|\ell^{-1}(k) \cap I|.$$

Applying this to $I_{\frac{1}{2}}$ and using (7.1) we have

$$b_{2k}(\Omega_{\frac{1}{2}}) = 2b_{2k}(G/B) = 2|\ell^{-1}(k)|.$$

Now consider the element $\mu \in S_{2n}$ given by the tuple

$$\mu = (2j, \ldots, j+1, 4j+2, j, \ldots 1, 4j+1, \ldots, 2j+1),$$

where in this expression $a \dots b$ denotes the integers between a and b in decreasing order. Then $\mu \notin I_{2n}$ since $\mu(2n) = n = 2j+1$. A straightforward application of (7.2) shows $\ell(\mu) = k$. As $\mu \in \ell^{-1}(k) - I_{2n}$ we have $|\ell^{-1}(k) \cap I_{2n}| < 2|\ell^{-1}(k)|$, which by the formulas above gives

$$b_{2k}(\Omega_{2n}) < b_{2k}(\Omega_{\frac{1}{2}}). \tag{7.6}$$

Applying Theorem 5.9, and using the vanishing of odd homology groups of Ω^I from Theorem 5.6, we have for any balanced ideal I that

$$b_{2k+1}(\mathcal{W}^I) = b_1(S) b_{2k}(\Omega^I).$$

Combining this with (7.6) we find $b_{2k+1}(\mathcal{W}_{2n}) < b_{2k+1}(\mathcal{W}_{\frac{1}{2}})$, and these manifolds are not homotopy equivalent. $\square$

7.6 The $\mathrm{PSL}_3\mathbb{C}$ case

In this final subsection, we consider $G = \mathrm{PSL}_3\mathbb{C}$ and give an alternative description of the limit set and domain of discontinuity in G/B for a G-Fuchsian group. This allows us to verify Conjecture 1.1 in this case. Chronologically, our study of this example preceded the other results of this paper, and indeed, the main results of Sections 5–6 resulted from an attempt to generalize aspects of the picture described below to other complex Lie groups.

For $G = \mathrm{PSL}_3\mathbb{C}$ there is unique balanced ideal $I = I_{\frac{1}{2}} = I_{1,2}$ in the Weyl group $W \simeq S_3$. Here $I = \{e, \alpha_1, \alpha_2\}$ where α_i are the simple root reflections, or in the permutation model, $I = \{(1,2,3), (2,1,3), (1,3,2)\}$. Since I is fixed we write $\Phi, \Lambda, \Omega, \mathcal{W}$, for the model thickening, limit set, domain, and quotient manifold, dropping the decoration by I from our notation.

Let $\varrho : \pi_1 S \to \mathrm{PSL}_3\mathbb{C}$ be a $\mathrm{PSL}_3\mathbb{C}$-Fuchsian representation, and in the rest of this section let $\mathcal{F} = G/B = \{(\ell, H) \,:\, \ell \subset H\}$ denote the flag variety. Let $X \subset \mathcal{F}$ denote the principal curve and $\widetilde{\varphi} = f_{\mathrm{PSL}_3\mathbb{C}} : \mathbb{P}^1_{\mathbb{C}} \to X$ its holomorphic parameterization. Let $Y \subset \mathbb{P}^2_{\mathbb{C}}$ denote the projection of the principal curve under the map $(\ell, H) \mapsto \ell$, and $\varphi : \mathbb{P}^1_{\mathbb{C}} \to Y$ the composition of $\widetilde{\varphi}$ with the same projection.

In what follows we regard an element $\ell \in \mathbb{P}^2_{\mathbb{C}}$ as a point in a complex surface, rather than as a 1-dimensional subspace of a 3-dimensional vector space.

There is a biholomorphic map $\mathbb{P}^2_{\mathbb{C}} \simeq \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$ which maps $\ell \in \mathbb{P}^2_{\mathbb{C}}$ to $p+q$ if ℓ lies on the tangent lines $T_{\varphi(p)}Y$ and $T_{\varphi(q)}Y$, and to $2p$ if $\ell = \varphi(p)$. Dually there is an identification $(\mathbb{P}^2_{\mathbb{C}})^*$ with $\mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$, where we regard $H \in (\mathbb{P}^2_{\mathbb{C}})^*$ as a projective line in $\mathbb{P}^2_{\mathbb{C}}$, and map H to the sum (with multiplicity) of the φ-preimages of its intersection points with Y.

Since $P_{1,n-1} = B$ for this group, following the discussion at the end of Section 7.2 we have the embedding $\mathcal{F} \hookrightarrow \mathbb{P}^2_{\mathbb{C}} \times (\mathbb{P}^2_{\mathbb{C}})^*$. Composing with the maps introduced above we then have $\mathcal{F} \hookrightarrow \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}}) \times \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$. It is easy to check that the principal curve $X \subset \mathcal{F}$ maps to the set $\{(2p, 2p) \,:\, p \in \mathbb{P}^1_{\mathbb{C}}\}$ and that $\widetilde{\varphi}(p) = (2p, 2p)$. Recall that the limit curve of ϱ is the circle $\widetilde{\varphi}(\mathbb{P}^1_{\mathbb{R}}) \subset \mathcal{F}$.

In order to give a geometric description of the limit set and domain of discontinuity, we further identify $\mathbb{P}^1_{\mathbb{C}}$ with the boundary at infinity of the 3-dimensional real hyperbolic space $\mathbb{H}^3$, for example using stereographic projection[4] to map $\mathbb{P}^1_{\mathbb{C}}$ to the unit sphere in $\mathbb{R}^3$ considered as the boundary of the unit ball model of $\mathbb{H}^3$. Let $\gamma_{p,q}$ denote the hyperbolic geodesic with ideal endpoints $p, q \in \mathbb{P}^1_{\mathbb{C}}$.

Lemma 7.8. *A point x in $\mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}}) \times \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$ lies in the image of $\mathcal{F}$ if and only if it satisfies one of the following mutually exclusive conditions:*

- *$x = (p+q, r+s)$ where p, q, r, s are pairwise distinct and the hyperbolic geodesics $\gamma_{p,q}$ and $\gamma_{r,s}$ intersect orthogonally, or*
- *$x = (2p, p+q)$ where $p \neq q$, or*
- *$x = (p+q, 2q)$ where $p \neq q$, or*
- *$x = (2p, 2p) \in X$.*

Proof. Suppose that $x = (\xi, \eta)$ corresponds to a flag (ℓ, H) where the divisors $\xi, \eta \in \mathrm{Sym}^2(\mathbb{P}^1_{\mathbb{C}})$ have a point in common, say p. By the construction of the embedding given above, this means

- The projective line $H \subset \mathbb{P}^2_{\mathbb{C}}$ passes through Y at $\varphi(p)$, and
- The tangent line $T_{\varphi(p)}Y$ contains ℓ.

Since $\ell \in H$, both $\varphi(p)$ and ℓ lie in $T_{\varphi(p)}Y \cap H$. Since distinct projective lines intersect in a single point, we have either $\ell = \varphi(p)$, in which case $\xi = 2p$, or $\mathcal{T}_{\varphi}(p)Y = H$, in which case $\eta = 2p$, or both.

This shows that x has one of the given forms, with the exception of the orthogonality condition in the first case. Hence we must show that for distinct p, q, r, s the geodesics $\gamma_{p,q}$ and $\gamma_{r,s}$ intersect orthogonally in $\mathbb{H}^3$ if and only if the corresponding pair of a point and projective line in $\mathbb{P}^2_{\mathbb{C}}$ form a flag, i.e. the projective line spanned by $\varphi(r)$ and $\varphi(s)$ is concurrent with the tangents $T_{\varphi(p)}Y$ and $T_{\varphi(q)}Y$. This can be done with an elementary explicit calculation, but we prefer to give a coordinate-free proof.

Given two points $p, q \in \mathbb{P}^1_{\mathbb{C}}$, the *half turn* $\tau_{p,q} : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ is the unique nontrivial holomorphic involution fixing p and q. Geometrically, $\tau_{p,q}$ is the extension to the ideal boundary of the isometry $\mathbb{H}^3 \to \mathbb{H}^3$ which rotates about $\gamma_{p,q}$ by angle π. Thus geodesics $\gamma_{p,q}$ and $\gamma_{r,s}$ intersection orthogonally if and only if $\{r, s\}$ is an orbit of $\tau_{p,q}$.

Given a pair of points $\{u, v\} \subset Y$, we can define a map $\widehat{\tau}_{u,v} : Y \to Y$ as follows: Let $H^* = T_uY \cap T_vY$, which is a point not on Y. The projective line joining H^* to $w \in Y$ intersects Y in a second point, which is $\widehat{\tau}_{u,v}(w)$. (See Figure 1.) Since this defines an

[4] More intrinsically, we could view $\mathbb{H}^3 \simeq \mathrm{SL}_2\mathbb{C}/\mathrm{SU}(2)$ as the space of hermitian forms on the vector space $H^0(Y, \mathcal{O}(1))$ that induce a given volume form—a space which is compactified by Y itself.

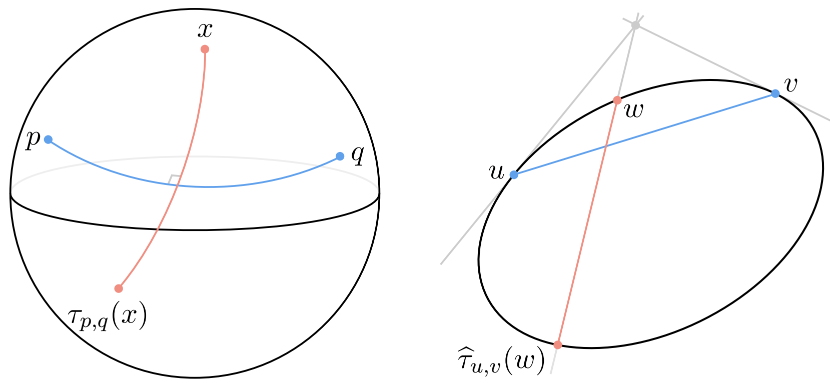

Figure 1: *Hyperbolic and projective models of a half turn on* $\mathbb{P}^1_{\mathbb{C}}$.

involutive, nontrivial holomorphic automorphism of Y fixing u and v, it is φ-conjugate to a half turn of $\mathbb{P}^1_{\mathbb{C}}$, i.e.

$$\widehat{\tau}_{u,v}(\varphi(t)) = \varphi(\tau_{p,q}(t)).$$

On the other hand, by definition of $\widehat{\tau}_{u,v}$ the points $\varphi(r), \varphi(s)$ form an orbit if and only if the projective line they span is concurrent with T_uY and T_vY. Hence the φ-conjugacy of $\widehat{\tau}$ and τ gives the desired equivalence between orthogonality and incidence. □

We now analyze the Kapovich-Leeb-Porti construction in terms of the *divisor model* of $\mathcal{F}$ given by the Lemma. First we note that the model thickening in this case is the union of the complex 1-dimensional Schubert varieties, $\Phi = X_{(2,1,3)} \cup X_{(1,3,2)}$, and it is easily checked that $X_{(2,1,3)} = \{(E_1, H) \,:\, H \in (\mathbb{P}^2_{\mathbb{C}})^*\}$ while $X_{(1,3,2)} = \{(\ell, E_2) \,:\, \ell \in \mathbb{P}^2_{\mathbb{C}}\}$. The corresponding description of Λ is that it consists of flags $\{(\ell, H)\}$ in which either $\ell \in \varphi(\mathbb{P}^1_{\mathbb{R}})$ or H is tangent to Y along $\varphi(\mathbb{P}^1_{\mathbb{R}})$. In terms of divisors, then, Λ consists of pairs (ξ, η) where either $\xi = 2p$ or $\eta = 2p$, for $p \in \mathbb{P}^1_{\mathbb{R}}$.

Let $\mathbb{H}_+, \mathbb{H}_-$ denote the connected components of $\mathbb{P}^1_{\mathbb{C}} - \mathbb{P}^1_{\mathbb{R}}$, and $X_\pm$ the compact Riemann surfaces that are the quotients of $\widetilde{\varphi}(\mathbb{H}_\pm) \subset Y$ by the ϱ-action of $\pi_1 S$. Considering each of the cases from Lemma 7.8, we find that $\Omega = \mathcal{F} - \Lambda$ can be described in the divisor model as $\Omega_0 \cup \widetilde{E}_+ \cup \widetilde{E}_+^* \cup \widetilde{E}_- \cup \widetilde{E}_-^*$ where

- $\Omega_0 = \{(p+q, r+s) \,:\, p \neq q,\ r \neq s\} \cap \mathcal{F}$,
- $\widetilde{E}_\pm = \{(2p, p+q) \,:\, p \in \mathbb{H}_\pm\}$, and
- $\widetilde{E}_\pm^* = \{(p+q, 2p) \,:\, p \in \mathbb{H}_\pm\}$.

Note that these sets are pairwise disjoint except for

$$\widetilde{E}_\pm \cap \widetilde{E}_\pm^* = \{(2p, 2p) \,:\, p \in \mathbb{H}_\pm\} = \widetilde{\varphi}(\mathbb{H}_\pm).$$

Now we arrive at the desired hyperbolic-geometric description of $\mathcal{W}$. Let $\varrho_0 : \pi_1 S \to \mathrm{PSL}_2\mathbb{R} < \mathrm{PSL}_2\mathbb{C} \simeq \mathrm{Isom}^+(\mathbb{H}^3)$ be the Fuchsian representation through which ϱ factors, or equivalently, so that $\widetilde{\varphi} : \mathbb{P}^1_{\mathbb{C}} \to X$ intertwines ϱ_0 acting on $\mathbb{P}^1_{\mathbb{C}}$ with ϱ acting on $\mathcal{F}$. Let N_0 denote the oriented orthonormal frame bundle of the quotient $\varrho_0(\pi_1 S)\backslash\mathbb{H}^3$ and define $N = N_0/(\mathbb{Z}/2 \times \mathbb{Z}/2)$ where $(i,j) \in \mathbb{Z}/2 \times \mathbb{Z}/2$ acts on an orthonormal frame $(v_1, v_2, v_3) \in T_x\mathbb{H}^3$ by

$$(v_1, v_2, v_3) \mapsto ((-1)^i v_1, (-1)^j v_2, (-1)^{i+j} v_3).$$

Since $\varrho_0(\pi_1 S)\backslash\mathbb{H}^3 \simeq S \times \mathbb{R}$, we have $N_0 \simeq S \times \mathbb{R} \times \mathrm{SO}(3)$ and $N \simeq S \times \mathbb{R} \times B$ where $B = \mathrm{SO}(3)/(\mathbb{Z}/2 \times \mathbb{Z}/2)$.

Theorem 7.9. *The quotient $\varrho(\pi_1 S)\backslash\Omega_0$ is diffeomorphic to N, and hence $\mathcal{W} = \varrho(\pi_1 S)\backslash\Omega$ is a compactification of N. The boundary of this compactification is the union of the four complex surfaces*

$$E_\pm := \varrho(\pi_1 S)\backslash\widetilde{E_\pm} \text{ and } E^*_\pm := \varrho(\pi_1 S)\backslash\widetilde{E^*_\pm},$$

each of which is biholomorphic to a $\mathbb{P}^1_{\mathbb{C}}$ bundle over X_+ or X_-, and which intersect only in the complex curves $E_+ \cap E^*_+ = X_+$ and $E_- \cap E^*_- = X_-$.

Proof. Using the divisor model, map $(p+q, r+s) \in \Omega_0$ to the positively oriented orthonormal frame (v_1, v_2, v_3) at $\gamma_{p,q} \cap \gamma_{r,s} \in \mathbb{H}^3$ such that v_1 is a unit vector along $\gamma_{p,q}$ and v_2 is a unit vector along $\gamma_{r,s}$. While there are two choices for each of v_1 and v_2, the result is a well-defined point in the quotient of the frame bundle of $\mathbb{H}^3$ by $\mathbb{Z}/2 \times \mathbb{Z}/2$. This map is easily seen to be a $\mathrm{PSL}_2\mathbb{C}$-equivariant, and both spaces have transitive, smooth $\mathrm{PSL}_2\mathbb{C}$ actions with the same isotropy, so it is a diffeomorphism. By equivariance it descends to the desired map $\varrho(\pi_1 S)\backslash\Omega_0 \to N$.

Lemma 7.8 describes Ω_0 as an open, dense, and ϱ-invariant subset of the cocompact domain of discontinuity Ω, hence $\mathcal{W}$ is a compactification of $\varrho(\pi_1 S)\backslash\Omega_0$. It remains to verify the given descriptions of the quotients of $\widetilde{E}_\pm$. We have already seen that $\widetilde{E}_\pm \cap \widetilde{E}^*_\pm = \widetilde{\varphi}(H_\pm)$ which has quotient $X_\pm$. To see that E_+ is a $\mathbb{P}^1_{\mathbb{C}}$ bundle over X_+, note first that $\widetilde{E}_+ \simeq \mathbb{H}_+ \times \mathbb{P}^1_{\mathbb{C}}$ by the map $(2p, p+q) \mapsto (p,q)$. Thus $\widetilde{E}_+$ is a trivial $\mathbb{P}^1_{\mathbb{C}}$ bundle over $\mathbb{H}_+$, and the projection $(2p, p+q) \mapsto p$ intertwines the ϱ-action on $\widetilde{E}_+$ with the ϱ_0-action on $\mathbb{H}_+$, and ϱ acts on $\widetilde{E}_+$ by a discontinuous group of bundle automorphisms. The quotient E_+ is therefore a locally trivial $\mathbb{P}^1_{\mathbb{C}}$ bundle over $\varrho_0(\pi_1 S)\backslash\mathbb{H}_+ \simeq \varrho(\pi_1 S)\backslash\widetilde{\varphi}(\mathbb{H}_+) = X_+$. The cases E_- and $E^*_\pm$ are handled similarly. $\square$

The decomposition of $\mathcal{W}$ described above is pictured schematically in Figure 2.

Since the oriented orthonormal frame bundle of 3-dimensional hyperbolic space is $\mathrm{PSL}_2\mathbb{C}$-equivariantly isomorphic to $\mathrm{PSL}_2\mathbb{C}$, Theorem 7.9 equivalently describes $\mathcal{W}$ as a compactification of the quotient $\varrho_0(\pi_1 S)\backslash\mathrm{PSL}_2\mathbb{C}/(\mathbb{Z}/2 \times \mathbb{Z}/2)$.

Finally, we will show that the divisor model and hyperbolic picture of $\mathcal{W}$ lead to a verification of Conjecture 1.1 (on the existence of a fiber bundle structure) in this case.

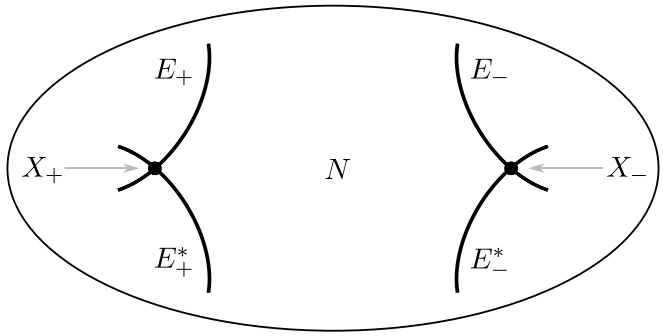

Figure 2: *Stratification of the* $\mathrm{PSL}_3\mathbb{C}$ *quotient manifold* $\mathcal{W}$ *consisting of the open stratum* N*, the* $\mathbb{P}^1_{\mathbb{C}}$*-bundles* $E_\pm$*,* $E_\pm^*$*, and the Riemann surfaces* $X_\pm$*.*

Such a fiber bundle structure is easy to construct for the open, dense set $N \subset \mathcal{W}$: There is a map from the frame bundle of $\mathbb{H}^3$ to $\mathbb{H}^2$ which composes the projection of the frame bundle to its base with the orthogonal projection from $\mathbb{H}^3$ to the totally geodesic $\mathbb{H}^2$ preserved by $\mathrm{PSL}_2\mathbb{R}$. This map is $(\mathbb{Z}/2 \times \mathbb{Z}/2)$-invariant and $\mathrm{PSL}_2\mathbb{R}$-equivariant; taking the quotient by $\mathbb{Z}/2 \times \mathbb{Z}/2$ and using the identification of Theorem 7.9 we obtain an induced $\mathrm{PSL}_2\mathbb{R}$-equivariant map

$$\widetilde{\pi} : \Omega_0 \to \mathbb{H}^2.$$

Taking a further quotient by $\varrho_0(\pi_1 S)$, a map $\pi : N \to S \simeq (\varrho_0(\pi_1 S)\backslash\mathbb{H}^2)$ is obtained. The identification of N with a product, $N \simeq S \times \mathbb{R} \times B$, can be made in such a way that the map π is simply projection onto the first factor.

To show that $\mathcal{W}$ is also a fiber bundle, we extend $\widetilde{\pi}$ and π to Ω and $\mathcal{W}$, respectively:

Theorem 7.10. *The map* $\widetilde{\pi} : \Omega_0 \to \mathbb{H}^2$ *extends to a proper* $\mathrm{PSL}_2\mathbb{R}$*-equivariant continuous map* $\widehat{\pi} : \Omega \to \mathbb{H}^2$*. Therefore,*

(i) Ω *has the structure of* $\mathrm{PSL}_2\mathbb{R}$*-equivariant continuous fiber bundle over* $\mathbb{H}^2$ *with fiber a compact topological space* F*,*

(ii) Ω *is homeomorphic to* $\mathbb{H}^2 \times F$*, and*

(iii) *The quotient manifold* $\mathcal{W} = \Gamma\backslash\Omega$ *is a continuous fiber bundle over* S *with fiber* F*.*

Proof. Statements (i)-(iii) are simple consequences of the existence of such a map $\widehat{\pi}$: Because $\mathbb{H}^2$ is a homogeneous space of $\mathrm{PSL}_2\mathbb{R}$, a continuous equivariant map from a $\mathrm{PSL}_2\mathbb{R}$-space to $\mathbb{H}^2$ is necessarily an equivariant locally trivial fibration. The fiber is compact by properness of $\widehat{\pi}$, so (i) follows. Since $\mathbb{H}^2$ is contractible this bundle is trivial, giving (ii).

Finally, using the equivariant structure of bundle $\widehat{\pi} : \Omega \to \mathbb{H}^2$ we can take the quotient by $\varrho_0(\pi_1 S)$ to obtain (iii).

Now we construct $\widehat{\pi}$. Let $\Omega' = \Omega - \Omega_0$, which is a closed set. Since we seek an extension of the map $\widetilde{\pi}$, it suffices to define $\widehat{\pi}$ on the set Ω', which in the divisor model consists of pairs of the form $(2p, p+q)$ or $(p+q, 2p)$ with $p \notin \mathbb{P}^1_{\mathbb{R}}$. Let $\Pi : \mathbb{P}^1_{\mathbb{C}} - \mathbb{P}^1_{\mathbb{R}} \to \mathbb{H}^2$ be the extension to the ideal boundary of orthogonal projection $\mathbb{H}^3 \to \mathbb{H}^2$; equivalently Π is the union of the natural $\mathrm{PSL}_2\mathbb{R}$-equivariant diffeomorphisms $\mathbb{H}_+ \to \mathbb{H}^2$ and $\mathbb{H}_- \to \mathbb{H}^2$. Define:

$$\widehat{\pi}(2p, p+q) = \Pi(p)$$
$$\widehat{\pi}(p+q, 2p) = \Pi(p)$$

This is evidently a continuous and $\mathrm{PSL}_2\mathbb{R}$-equivariant map $\Omega' \to \mathbb{H}^2$, since the map Π itself has these properties and the two definitions above agree on their common domain $\{(2p, 2p) \,:\, p \in \mathbb{P}^1_{\mathbb{C}} - \mathbb{P}^1_{\mathbb{R}}\}$.

It remains to show that $\widehat{\pi}$ is continuous on the entire domain Ω, and that it is proper. Both will follow by elementary geometric arguments.

For continuity, since Ω' is closed it suffices to consider a sequence $\omega_n \in \Omega_0$ converging to $\omega_\infty \in \Omega'$ and to show $\widetilde{\pi}(\omega_n) \to \widehat{\pi}(\omega_\infty)$. We suppose the limit point has the form $\omega_\infty = (2p, p+q)$ with $p \in \mathbb{H}_+$, the argument in the other cases being completely analogous. Since $\omega_n \in \Omega_0$, we can write $\omega_n = (p_n + p_n', p_n'' + q_n)$ with each of the sequences $\{p_n\}, \{p_n'\}, \{p_n''\}$ converging to p, and with $q_n \to q$. Recalling the construction of $\widetilde{\pi}$ and the map from the frame bundle to Ω_0 from the proof of Theorem 7.9, we see that $\widetilde{\pi}(\omega_n)$ is the orthogonal projection to $\mathbb{H}^2$ of the point $\gamma_{p_n, p_n'} \cap \gamma_{p_n'', q_n} \in \mathbb{H}^3$

Consider the disk $D \subset \mathbb{H}_+$ of radius ϵ centered at p with respect to the Poincaré metric of $\mathbb{H}_+$. The orthogonal projection to $\mathbb{H}^2$ of any geodesic in $\mathbb{H}^3$ with ideal endpoints in D is contained in the ϵ-disk centered at $\Pi(p) = \widehat{\pi}(\omega_\infty)$. For large enough n we have $p_n, p_n', p_n'' \in D$, and $\widetilde{\pi}(\omega_n)$ is the projection to $\mathbb{H}^2$ of a point on $\gamma_{p_n, p_n'}$, hence $d_{\mathbb{H}^2}(\widetilde{\pi}(\omega_n), \widehat{\pi}(\omega_\infty)) < \epsilon$. Thus $\widetilde{\pi}(\omega_n) \to \widehat{\pi}(\omega_\infty)$ as $n \to \infty$, and $\widehat{\pi}$ is continuous.

To see that $\widehat{\pi}$ is proper, we consider a compact exhaustion of Ω constructed by taking complements of small open neighborhoods of Λ. Recall Λ consists of divisor pairs of the form $(2p, p+q)$ or $(p+q, 2p)$ where p lies on $\mathbb{P}^1_{\mathbb{R}}$. Fix an auxiliary metric on $\mathbb{P}^1_{\mathbb{C}}$ and define $N_\varepsilon(\Lambda)$ to consist of divisor pairs $(p+q, r+s)$ in which there is a disk of radius ϵ in $\mathbb{P}^1_{\mathbb{C}}$ with center in $\mathbb{P}^1_{\mathbb{R}}$ which contains at least three of the points p, q, r, s.

Fix a basepoint x_0 in $\mathbb{H}^2$ (which we could take to be the origin in the unit ball model of $\mathbb{H}^3$). Then for each $R > 0$ there exists $\varepsilon = \varepsilon(R) > 0$ such that if $y \in \mathbb{H}^3$ lies in the hyperbolic convex hull of a disk on $\mathbb{P}^1_{\mathbb{C}}$ of radius ε, then $d_{\mathbb{H}^3}(x_0, y) > R$. That is, a half space in $\mathbb{H}^3$ bounded by a sufficiently small circle is far from x_0.

We claim that if $\omega \in N_\varepsilon(\Lambda) \cap \Omega$, then $\widehat{\pi}(\omega)$ lies in such a half-space, and thus is far from x_0 for ε small enough. To see this, first consider $\omega \in N_\varepsilon(\Lambda) \cap \Omega_0$ which we can write as $\omega = (p+q, r+s)$ with p, q, r, s distinct, and so that p, q, r lie in an ε-disk D which is centered on $\mathbb{P}^1_{\mathbb{R}}$. Let B be the half-space in $\mathbb{H}^3$ with ideal boundary D; note B is invariant by reflection in $\mathbb{H}^2$ and D is invariant by inversion in $\mathbb{P}^1_{\mathbb{R}}$. Then $\widehat{\pi}(\omega) = \widetilde{\pi}(\omega)$ is the orthogonal projection to $\mathbb{H}^2$ of a point $x \in \gamma_{p,q} \subset \mathbb{H}^3$. Since both x and its reflection $\overline{x}$ in $\mathbb{H}^2$ lie in B, so does the segment joining them. The intersection of this segment with $\mathbb{H}^2$ is the orthogonal projection of x to $\mathbb{H}^2$, which is $\widehat{\pi}(\omega)$, so $\widehat{\pi}(\omega) \in B$.

The remaining case is that $\omega \in \Omega'$, in which case we can write $\omega = (2p, p+q)$ or $\omega = (p+q, 2p)$, with p in an ε-disk D of the type considered above. Then $\widehat{\pi}(\omega) = \Pi(p)$, and $\Pi(p) \in B$ because it lies on the geodesic $\gamma_{p,\overline{p}}$, where $\overline{p}$ is the inversion of p in $\mathbb{P}^1_{\mathbb{R}}$, and $p, \overline{p} \in D$.

Now if $\omega_n \in \Omega$ satisfies $\omega_n \to \infty$, then for each $R > 0$ we have for all sufficiently large n that $\omega_n \in N_{\varepsilon(R)}(\Lambda)$. The argument above shows $d_{\mathbb{H}^2}(x_0, \widehat{\pi}(\omega_n)) > R$ for such n. Thus $\widehat{\pi}(\omega_n) \to \infty$ in $\mathbb{H}^2$, and $\widehat{\pi}$ is proper. $\square$

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