

Coarse and fine geometry of the Thurston metric

David Dumas, Anna Lenzhen, Kasra Rafi, and Jing Tao

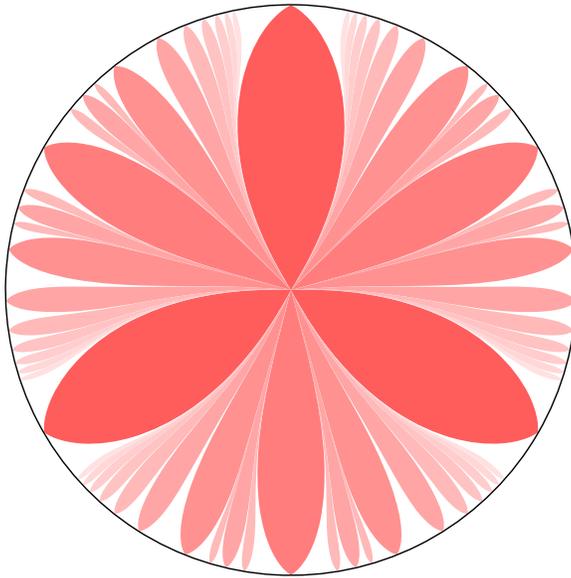


Figure 0: *In-envelopes in Teichmüller space; see Section 5.2.*

1 Introduction

In this paper we study the geometry of the Thurston metric on the Teichmüller space $\mathcal{T}(S)$ of hyperbolic structures on a surface S . Some of our results on the coarse geometry of this metric apply to arbitrary surfaces S of finite type; however, we focus particular attention on the case where the surface is a once-punctured torus, $S = S_{1,1}$. In that case, our results provide a detailed picture of the infinitesimal, local, and global behavior of the geodesics of the Thurston metric on $\mathcal{T}(S_{1,1})$, as well as an analogue of Royden's theorem (cf. [Roy71]).

Thurston's metric

Recall that Thurston's metric $d_{\text{Th}} : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathbf{R}$ is defined by

$$d_{\text{Th}}(X, Y) = \sup_{\alpha} \log \left(\frac{\ell_{\alpha}(Y)}{\ell_{\alpha}(X)} \right) \quad (1)$$

where the supremum is over all simple closed curves α in S and $\ell_\alpha(X)$ denotes the hyperbolic length of the curve α in X . This function defines a forward-complete asymmetric Finsler metric, introduced by Thurston in [Thu86b]. In the same paper, Thurston introduced two key tools for understanding this metric which will be essential in what follows: *stretch paths* and *maximally stretched laminations*.

The maximally stretched lamination $\Lambda(X, Y)$ is a recurrent geodesic lamination which is defined for any pair of distinct points $X, Y \in \mathcal{T}(S)$. Typically $\Lambda(X, Y)$ is just a simple curve, in which case that curve uniquely realizes the supremum defining d_{Th} . In general $\Lambda(X, Y)$ can be a more complicated lamination that is constructed from limits of sequences of curves that asymptotically realize the supremum. The precise definition is given in Section 2.4 (or [Thu86b, Section 8], where the lamination is denoted $\mu(X, Y)$).

Stretch paths are geodesic rays constructed from certain decompositions of the surface into ideal triangles. More precisely, given a hyperbolic structure $X \in \mathcal{T}(S)$ and a complete geodesic lamination λ one obtains a parameterized stretch path, $\text{stretch}(X, \lambda, \cdot) : \mathbf{R}^+ \rightarrow \mathcal{T}(S)$, with $\text{stretch}(X, \lambda, 0) = X$ and which satisfies

$$d_{\text{Th}}(\text{stretch}(X, \lambda, s), \text{stretch}(X, \lambda, t)) = t - s$$

for all $s, t \in \mathbf{R}$ with $s < t$.

Thurston showed that there also exist geodesics in $\mathcal{T}(S)$ that are concatenations of segments of stretch paths along different geodesic laminations. The abundance of such “chains” of stretch paths is sufficient to show that d_{Th} is a geodesic metric space, and also that it is not uniquely geodesic—some pairs of points are joined by more than one geodesic segment.

Envelopes

The first problem we consider is to quantify the failure of uniqueness for geodesic segments with given start and end points. For this purpose we consider the set $E(X, Y) \subset \mathcal{T}(S)$ that is the union of all geodesics from X to Y . We call this the *envelope* (from X to Y).

Based on Thurston’s construction of geodesics from chains of stretch paths, it is natural to expect that the envelope would admit a description in terms of the maximally stretched lamination $\Lambda(X, Y)$ and its completions. We focus on the punctured torus case, because here the set of completions is always finite.

In fact, a recurrent lamination on $S_{1,1}$ (such as $\Lambda(X, Y)$, for any $X \neq Y \in \mathcal{T}(S_{1,1})$) is either

- (a) A simple closed curve,
- (b) The union of a simple closed curve and a spiral geodesic, or
- (c) A measured lamination with no closed leaves

These possibilities are depicted in Figure 1.

We show that the geodesic from X to Y is unique when $\Lambda(X, Y)$ is of type (b) or (c), and when it has type (a) the envelope has a simple, explicit description. More precisely, we have:

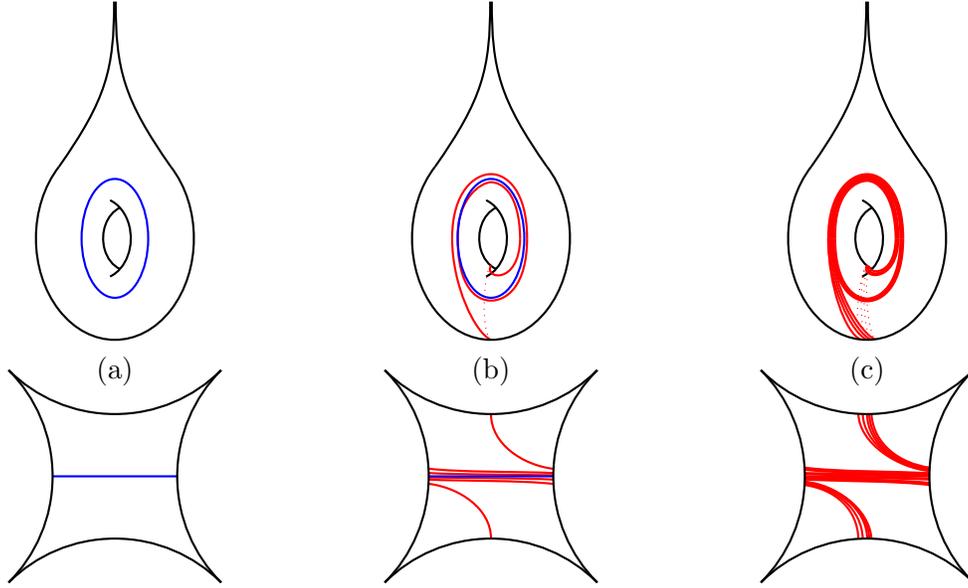


Figure 1: *The three types of recurrent laminations on $S_{1,1}$.*

Theorem 1.1 (Structure of envelopes for the punctured torus).

- (i) *For any $X, Y \in \mathcal{T}(S_{1,1})$, the envelope $E(X, Y)$ is a compact set.*
- (ii) *$E(X, Y)$ varies continuously in the Hausdorff topology as a function of X and Y .*
- (iii) *If $\Lambda(X, Y)$ is not a simple closed curve, then $E(X, Y)$ is a segment on a stretch path (which is therefore the unique geodesic from X to Y).*
- (iv) *If $\Lambda(X, Y) = \alpha$ is a simple closed curve, then $E(X, Y)$ is a geodesic quadrilateral with X and Y as opposite vertices. Each edge of the quadrilateral is a stretch path along the completion of a recurrent geodesic lamination properly containing α .*

In the course of proving the theorem above, we write explicit equations for the edges of the quadrilateral-type envelopes in terms of Fenchel-Nielsen coordinates (see (14)–(15)).

This theorem also highlights a distinction between two cases in which the d_{Th} -geodesic from X to Y is unique—the cases (b) and (c) discussed above. In case (b) the geodesic to Y is unique but some initial segment of it can be chained with another stretch path and remain geodesic: The boundary of a quadrilateral-type envelope from X with maximal stretch lamination α furnishes an example of this. In case (c), however, a geodesic that starts along the stretch path from X to Y is entirely contained in that stretch path (see Proposition 5.2).

Short curves

Returning to the case of an arbitrary surface S of finite type, in section 3 we establish results on the coarse geometry of Thurston metric geodesic segments. This study is similar in spirit to the one of Teichmüller geodesics in [Raf05], in that we seek to determine whether or not a simple curve α becomes short along a geodesic from X to Y . As in that case, a key quantity to consider is the amount of twisting along α from X to Y , denoted $d_\alpha(X, Y)$ and defined in Section 2.3.

For curves that *interact* with the maximally stretched lamination $\Lambda(X, Y)$, meaning they belong to the lamination or intersect it essentially, we show that becoming short on a geodesic with endpoints in the thick part of $\mathcal{T}(S)$ is equivalent to the presence of large twisting:

Theorem 1.2. *Let X, Y lie in the thick part of $\mathcal{T}(S)$ and let α be a simple curve on S that interacts with $\Lambda(X, Y)$. The minimum length ℓ_α of α along any Thurston metric geodesic from X to Y satisfies*

$$\frac{1}{\ell_\alpha} \text{Log} \frac{1}{\ell_\alpha} \stackrel{*}{\underset{\mp}{\approx}} d_\alpha(X, Y)$$

with implicit constants that are independent of α , and where $\text{Log}(x) = \min(1, \log(x))$.

Here $\stackrel{*}{\underset{\mp}{\approx}}$ means equality up to an additive and multiplicative constant; see Section 2.1. The theorem above and additional results concerning length functions along geodesic segments are combined in Theorem 3.1.

In Section 4 we specialize once again to the Teichmüller space of the punctured torus in order to say more about the coarse geometry of Thurston geodesics. Here every simple curve interacts with every lamination, so Theorem 1.2 is a complete characterization of short curves in this case. Furthermore, in this case we can determine the order in which the curves become short.

To state the result, we recall that the pair of points $X, Y \in \mathcal{T}(S_{1,1})$ determine a geodesic in the dual tree of the Farey tessellation of $\mathbf{H}^2 \simeq \mathcal{T}(S_{1,1})$. Furthermore, this path distinguishes an ordered sequence of simple curves—the *pivots*—and each pivot has an associated *coefficient*. These notions are discussed further in Section 4.

We show that pivots for X, Y and short curves on a d_{Th} -geodesic from X to Y coarsely coincide in an order-preserving way, once again assuming that X and Y are thick:

Theorem 1.3. *Let $X, Y \in \mathcal{T}(S_{1,1})$ lie in the thick part, and let $\mathcal{G}: I \rightarrow \mathcal{T}(S_{1,1})$ be a geodesic of d_{Th} from X to Y . Let ℓ_α denote the minimum of $\ell_\alpha(\mathcal{G}(t))$ for $t \in I$. We have:*

- (i) *If α is short somewhere in \mathcal{G} , then α is a pivot.*
- (ii) *If α is a pivot with large coefficient, then α becomes short somewhere in \mathcal{G} .*
- (iii) *If both α and β become short in \mathcal{G} , then they do so in disjoint intervals whose ordering in I agrees with that of α, β in $\text{Pivot}(X, Y)$.*
- (iv) *There is an a priori upper bound on ℓ_α for $\alpha \in \text{Pivot}(X, Y)$.*

In this statement, various constants have been suppressed (such as those required to make *short* and *large* precise). We show that all of the constants can be taken to be independent of X and Y , and the full statement with these constants is given as Theorem 4.3 below.

We have already seen that there may be many Thurston geodesics from X to Y , and due to the asymmetry of the metric, reversing parameterization of a geodesic from X to Y does not give a geodesic from Y to X . On the other hand, the notion of a pivot is symmetric in X and Y . Therefore, by comparing the pivots to the short curves of an arbitrary Thurston geodesic, Theorem 4.3 establishes a kind of symmetry and uniqueness for the *combinatorics* of Thurston geodesic segments, despite the failure of symmetry or uniqueness for the geodesics themselves.

Rigidity

A Finsler metric on $\mathcal{T}(S)$ gives each tangent space $T_X\mathcal{T}(S)$ the structure of a normed vector space. Royden showed that for the Teichmüller metric, this normed vector space uniquely determines X up to the action of the mapping class group [Roy71]. That is, the tangent spaces are isometric (by a linear map) if and only if the hyperbolic surfaces are isometric.

We establish the corresponding result for the Thurston’s metric on $\mathcal{T}(S_{1,1})$ and its corresponding norm $\|\cdot\|_{\text{Th}}$ (the *Thurston norm*) on the tangent bundle.

Theorem 1.4. *Let $X, Y \in \mathcal{T}(S_{1,1})$. Then there exists an isometry of normed vector spaces*

$$(T_X\mathcal{T}(S_{1,1}), \|\cdot\|_{\text{Th}}) \rightarrow (T_Y\mathcal{T}(S_{1,1}), \|\cdot\|_{\text{Th}})$$

if and only if X and Y are in the same orbit of the extended mapping class group.

The idea of the proof is to recognize lengths and intersection numbers of curves on X from features of the unit sphere in $T_X\mathcal{T}(S)$. Analogous estimates for the shape of the cone of lengthening deformations of a hyperbolic one-holed torus were established in [Gué15]. In fact, Theorem 1.4 was known to Guéritaud and can be derived from those estimates [Gué16]. We present a self-contained argument that does not use Guéritaud’s results directly, though [Gué15, Section 5.1] provided inspiration for our approach to the infinitesimal rigidity statement.

A local rigidity theorem can be deduced from the infinitesimal one, much as Royden did in [Roy71].

Theorem 1.5. *Let U be a connected open set in $\mathcal{T}(S_{1,1})$, considered as a metric space with the restriction of d_{Th} . Then any isometric embedding $(U, d_{\text{Th}}) \rightarrow (\mathcal{T}(S_{1,1}), d_{\text{Th}})$ is the restriction to U of an element of the extended mapping class group.*

Intuitively, this says that the quotient of $\mathcal{T}(S_{1,1})$ by the mapping class group is “totally unsymmetric”; each ball fits into the space isometrically in only one place. Of course, applying Theorem 1.5 to $U = \mathcal{T}(S_{1,1})$ we have the immediate corollary

Corollary 1.6. *Every isometry of $(\mathcal{T}(S_{1,1}), d_{\text{Th}})$ is induced by an element of the extended mapping class group, hence the isometry group is isomorphic to $\text{PGL}(2, \mathbf{Z})$.*

Here we have used the usual identification of the mapping class group of $S_{1,1}$ with $\mathrm{GL}(2, \mathbf{Z})$, whose action on $\mathcal{T}(S_{1,1})$ factors through the quotient $\mathrm{PGL}(2, \mathbf{Z})$.

The analogue of Corollary 1.6 for Thurston’s metric on higher-dimensional Teichmüller spaces was established by Walsh in [Wal14] using a characterization of the horofunction compactification of $\mathcal{T}(S)$. Walsh’s argument does not apply to the punctured torus, however, because it relies on Ivanov’s characterization (in [Iva97]) of the automorphism group of the curve complex (a result which does not hold for the punctured torus).

Passing from the infinitesimal (i.e. norm) rigidity to local or global statements requires some preliminary study of the smoothness of the Thurston norm. In Section 6.1 we show that the norm is locally Lipschitz continuous on $T\mathcal{T}(S)$ for any finite type hyperbolic surface S . By a recent result of Matveev-Troyanov [MT16], it follows that any d_{Th} -preserving map is differentiable with norm-preserving derivative. This enables the key step in the proof of Theorem 1.5, where Theorem 1.4 is applied to the derivative of the isometry.

Additional notes and references

In addition to Thurston’s paper [Thu86b], an exposition of Thurston’s metric and a survey of its properties can be found in [PT07]. Prior work on the coarse geometry of the Thurston metric on Teichmüller space and its geodesics can be found in [CR07] [LRT12] [LRT15]. The notion of the maximally stretched lamination for a pair of hyperbolic surfaces has been generalized to higher-dimensional hyperbolic manifolds [Kas09] [GK] and to vector fields on \mathbf{H}^2 equivariant for convex cocompact subgroups of $\mathrm{PSL}(2, \mathbf{R})$ [DGK16].

Acknowledgments

The authors thank the American Institute of Mathematics for hosting the workshop “Lipschitz metric on Teichmüller space” and the Mathematical Sciences Research Institute for hosting the semester program “Dynamics on Moduli Spaces of Geometric Structures” where some of the work reported here was completed. The authors gratefully acknowledge grant support from NSF DMS-0952869 (DD), NSF DMS-1611758 (JT), NSERC RGPIN 435885 (KR). The authors also thank François Guéritaud for helpful conversations related to this work, and specifically for suggesting the statement of Theorem 6.5.

2 Background

2.1 Approximate comparisons

We use the notation $a \stackrel{*}{\approx} b$ to mean that quantities a and b are equal up to a uniform multiplicative error, i.e. that there exists a positive constant K such that $K^{-1}a \leq b \leq Ka$. Similarly $a \stackrel{*}{\preceq} b$ means that $a \leq Kb$ for some K .

The analogous relations up to additive error are $a \stackrel{\pm}{\approx} b$, meaning that there exists C such that $a - C \leq b \leq a + C$, and $a \stackrel{\pm}{\preceq} b$ which means $a \leq b + C$ for some C .

For equality up to both multiplicative and additive error, we write $a \stackrel{*}{\asymp} b$. That is, $a \stackrel{*}{\asymp} b$ means that there exist constants K, C such that $K^{-1}a - C \leq b \leq Ka + C$.

Unless otherwise specified, the implicit constants depend only on the topological type of the surface S . When the constants depend on the Riemann surface X , we use the notation $\stackrel{*}{\asymp}_X$ and $\stackrel{\pm}{\asymp}_X$ instead.

For functions f, g of a real variable x we write $f \sim g$ to mean that $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

2.2 Curves and laminations

We first record some standard definitions and set our conventions for objects associated to a connected and oriented surface S . A *multicurve* is a 1-manifold on S defined up to homotopy such that no connected component is homotopic to a point, a puncture, or boundary of S . A connected multicurve will just be called a *curve*. Note that with our definition, there are no curves on the two- or three-punctured sphere, so we will ignore those cases henceforth. The *geometric intersection number* $i(\alpha, \beta)$ between two curves is the minimal number of intersections between representatives of α and β . If we fix a hyperbolic metric on S , then $i(\alpha, \beta)$ is just the number of intersections between the geodesic representative of α with the geodesic representative of β . For any curve α on S , we denote by D_α the left Dehn twist about α . This is well-defined after fixing an orientation for S .

Fix a hyperbolic metric on S . A *geodesic lamination* λ on S is a closed subset which is a disjoint union of simple complete geodesics. These geodesics are called the *leaves* of λ . Two different hyperbolic metrics on S determine canonically isomorphic spaces of geodesic laminations, so the space of geodesic laminations $\mathcal{GL}(S)$ depends only on the topology of S . This is a compact metric space equipped with the metric of Hausdorff distance on closed sets. The closure of the set of multicurves in $\mathcal{GL}(S)$ is the set of *recurrent* laminations (or *chain-recurrent*, in Thurston's original terminology).

We will call a geodesic lamination *maximal-recurrent* if it is recurrent and not properly contained in another recurrent lamination. A geodesic lamination is *complete* if its complementary regions in S are ideal triangles. When S is a closed surface, then a maximal-recurrent lamination is complete. If S has punctures, then the complementary regions of a maximal-recurrent lamination are either ideal triangles or punctured bigons. The *stump* of a geodesic lamination (in the terminology of [Thé07]) is its maximal compactly-supported sublamination that admits a transverse measure of full support.

2.3 Twisting

There are several notions of twisting which we will define below. While these notions are defined for different classes of objects, in cases where several of the definitions apply, they are equal up to an additive constant.

Let A be an annulus. Fix an orientation of the core curve α of A . For any simple arc ω in A with endpoints on different components of ∂A , we orient ω so that the algebraic intersection number $\omega \cdot \alpha$ is equal to one. This allows us to define the algebraic intersection number $\omega \cdot \omega'$

between any two such arcs which is independent of the choice of the orientation of α . With our choice, we always have $\omega \cdot D_\alpha(\omega) = 1$, where as above D_α denotes the left Dehn twist about α .

Now let S be a surface and α is a simple closed curve on S . Let $\widehat{S} \rightarrow S$ be the covering space associated to $\pi_1(\alpha) < \pi_1(S)$. Then \widehat{S} has a natural Gromov compactification that is homeomorphic to a closed annulus. By construction, the core curve $\widehat{\alpha}$ of this annulus maps homeomorphically to α under this covering map.

Let λ and λ' be two geodesic laminations (possibly curves) on S , both intersecting α . We define their (signed) twisting relative to α as $\text{twist}_\alpha(\lambda, \lambda') = \min \widehat{\omega} \cdot \widehat{\omega}'$, where $\widehat{\omega}$ is a lift of a leaf of λ and $\widehat{\omega}'$ is a lift of a leaf of λ' , with both lifts intersecting $\widehat{\alpha}$, and the minimum is taken over all such leaves and their lifts. Note that for any two such lifts ω and ω' (still intersecting $\widehat{\alpha}$) the quantity $\widehat{\omega} \cdot \widehat{\omega}'$ exceeds $\text{twist}_\alpha(\lambda, \lambda')$ by at most 1.

Next we define the twisting of two hyperbolic metrics X and Y on S relative to α . Let \widehat{X}, \widehat{Y} denote the lifts of these hyperbolic structures to \widehat{S} . Using the hyperbolic structure \widehat{X} , choose a geodesic $\widehat{\omega}$ that is orthogonal to the geodesic in the homotopy class of $\widehat{\alpha}$. Let $\widehat{\omega}'$ be a geodesic constructed similarly from \widehat{Y} . We set $\text{twist}_\alpha(X, Y) = \min \widehat{\omega} \cdot \widehat{\omega}'$, where the minimum is taken over all possible choices for $\widehat{\omega}$ and $\widehat{\omega}'$. Similar to the previous case, this minimum differs from the intersection number $\widehat{\omega} \cdot \widehat{\omega}'$ for a particular pair of choices by a most 1.

Finally, we define $\text{twist}_\alpha(X, \lambda)$, the twisting of a lamination λ about a curve α on X . This is defined if λ contains a leaf that intersects α . Let $\widehat{\omega}$ be a geodesic of \widehat{X} orthogonal to the geodesic homotopic to $\widehat{\alpha}$. Let ω' be any leaf of λ intersecting α , and let $\widehat{\omega}'$ be a lift of this leaf to \widehat{X} which intersects $\widehat{\alpha}$. Then $\text{twist}_\alpha(X, \lambda) = \min \widehat{\omega} \cdot \widehat{\omega}'$, with the minimum taken over all choices of ω' , $\widehat{\omega}'$, and $\widehat{\omega}$. The same quantity can be defined using the universal cover $\widetilde{X} \cong \mathbf{H}$. Let $\widetilde{\alpha}$ be lift of α and let $\widetilde{\omega}$ be a lift of a leaf of λ intersecting $\widetilde{\alpha}$. Let L be the length of the orthogonal projection of $\widetilde{\omega}$ to $\widetilde{\alpha}$ and let ℓ be the length of the geodesic representative of α on X . Then, by [Min96], $|\text{twist}_\alpha(X, \lambda)| \stackrel{\pm}{\leq} L/\ell$.

Each type of twisting defined above is *signed*. In some cases the absolute value of the twisting is the relevant quantity; we use the notation $d_\alpha(\cdot, \cdot) = |\text{twist}_\alpha(\cdot, \cdot)|$ for the corresponding *unsigned twisting* in each case.

2.4 Teichmüller space and the Thurston metric

Let $\mathcal{T}(S)$ be the Teichmüller space of marked finite-area hyperbolic surfaces homeomorphic to S . The space $\mathcal{T}(S)$ is homeomorphic to $\mathbf{R}^{6g-6+2n}$ if S has genus g and n punctures. Given $X \in \mathcal{T}(S)$ and a curve α on S , we denote by $\ell_\alpha(X)$ the length of the geodesic representative of α on X .

The Margulis constant is the smallest constant $\epsilon_M > 0$ such that on any hyperbolic surface X , if two curves α and β both have lengths less than ϵ_M , then they are disjoint. For any $\epsilon > 0$, we will denote by $\mathcal{T}_\epsilon(S)$ the set of points in $\mathcal{T}(S)$ on which every curve has length at least ϵ ; this is the ϵ -*thick part* of Teichmüller space.

For a pair of points $X, Y \in \mathcal{T}(S)$, in the introduction we defined the quantity

$$d_{\text{Th}}(X, Y) = \sup_{\alpha} \log \frac{\ell_{\alpha}(Y)}{\ell_{\alpha}(X)}$$

where the supremum is taken over all simple curves. Another measure of the difference of hyperbolic structures, in some ways dual to this length ratio, is

$$L(X, Y) = \inf_f \log L_f$$

where L_f is the Lipschitz constant, and where the infimum is taken over Lipschitz maps in the preferred homotopy class. Thurston showed:

Theorem 2.1. *For all $X, Y \in \mathcal{T}(S)$ we have $d_{\text{Th}}(X, Y) = L(X, Y)$, and this function is an asymmetric metric, i.e. it is positive unless $X = Y$ and it obeys the triangle inequality.*

Thurston also showed that the infimum Lipschitz constant is realized by a homeomorphism from X to Y . Any map which realizes the infimum is called *optimal*. Further, there is a unique recurrent lamination $\Lambda(X, Y)$, called the *maximally stretched lamination*, such that any optimal map from X to Y multiplies arc length along this lamination by a constant factor of $e^{L(X, Y)}$.

The same lamination $\Lambda(X, Y)$ can be characterized in terms of length ratios as well: Any sequence of simple curves that asymptotically realizes the supremum in the definition of $d_{\text{Th}}(X, Y)$ has a subsequence converging, in the Hausdorff topology, to a geodesic lamination. Then $\Lambda(X, Y)$ is the union of all laminations obtained this way.

The length ratio for simple curves extends continuously to the space of projective measured lamination, which is compact. Therefore, the length-ratio supremum is always realized by some measured lamination. Any measured lamination that realizes the supremum has support contained in the stump of $\Lambda(X, Y)$.

Suppose a parameterized path $\mathcal{G}: [0, d] \rightarrow \mathcal{T}(S)$ is a geodesic from X to Y (parameterized by unit speed). Then the following holds: for any $s, t \in [0, d]$ with $s < t$ and for any arc ω contained in the geometric realization of $\Lambda(X, Y)$ on X , the arc length of ω is stretched by a factor of e^{t-s} under any optimal map from $\mathcal{G}(s)$ to $\mathcal{G}(t)$. We will sometimes denote $\Lambda(X, Y)$ by $\lambda_{\mathcal{G}}$

Given any complete geodesic lamination λ , there is an embedding $s_{\lambda}: \mathcal{T}(S) \rightarrow \mathbf{R}^N$ by the *shearing coordinates* relative to λ , where $N = \dim \mathcal{T}(S)$. The image of this embedding is an open convex cone. Details of the construction of this embedding can be found in [Bon96] and [Thu86b, Section 9], and in the next section we will define s_{λ} carefully when $S = S_{1,1}$ is the once-punctured torus and λ has finitely many leaves. Postponing that discussion for a moment, we use the shearing coordinates to describe the stretch paths mentioned in the introduction.

Given any $X \in \mathcal{T}(S)$, a complete geodesic lamination λ , and $t \geq 0$, let $\text{stretch}(X, \lambda, t)$ be the unique point in $\mathcal{T}(S)$ such that

$$s_{\lambda}(\text{stretch}(X, \lambda, t)) = e^t s_{\lambda}(X).$$

Letting t vary, we have that $\text{stretch}(x, \lambda, t)$ is a parameterized path in $\mathcal{T}(S)$ that maps to an open ray from the origin in \mathbf{R}^N under the shearing coordinates.

Thurston showed that the path $t \mapsto \text{stretch}(X, \lambda, t)$ is a geodesic ray in $\mathcal{T}(S)$, with λ the maximally stretched lamination for any pair of points. This is called a *stretch path along λ from X* . Given $X, Y \in \mathcal{T}(S)$, if $\Lambda(X, Y)$ is a complete lamination, then the only Thurston geodesic from X to Y is the stretch path along $\Lambda(X, Y)$. When $\Lambda(X, Y)$ is not complete, then one can still connect X to Y by a concatenation of finitely many stretch paths using different completions of $\Lambda(X, Y)$. In particular, this implies that $\mathcal{T}(S)$ equipped with the Thurston metric is a geodesic metric space.

Denote by $\bar{d}_{\text{Th}}(X, Y) = \max\{d_{\text{Th}}(X, Y), d_{\text{Th}}(Y, X)\}$. The topology of $\mathcal{T}(S)$ is compatible with \bar{d}_{Th} , so by $X_i \rightarrow X$ we will mean $\bar{d}_{\text{Th}}(X_i, X) \rightarrow 0$. By the Hausdorff distance on closed sets in $\mathcal{T}(S)$ we will mean with respect to the metric \bar{d}_{Th} .

2.5 Shearing coordinates for $\mathcal{T}(S_{1,1})$

Let \mathbf{H} denote the upper half plane model of the hyperbolic plane, with ideal boundary $\partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$. Two distinct points $x, y \in \partial\mathbf{H}$ determine a geodesic $[x, y]$ and three distinct points $x, y, z \in \partial\mathbf{H}$ determine an ideal triangle $\Delta(x, y, z)$. Recall that an ideal triangle in \mathbf{H} has a unique inscribed circle which is tangent to all three sides of the triangle. Each tangency point is called the *midpoint* of the side.

Let $\gamma = [\gamma^+, \gamma^-]$ be a geodesic in \mathbf{H} . Suppose two ideal triangles Δ and Δ' lie on different sides of γ . We allow the possibility that γ is an edge of Δ or Δ' (or both). Suppose Δ is asymptotic to γ^+ and the Δ' is asymptotic to γ^- . Let m be the midpoint along the side of Δ closest to γ . The pair γ^+ and m determine a horocycle that intersects γ at a point p . Let m' and p' be defined similarly using Δ' and γ^- . We say p' is to the left of p (relative to Δ and Δ') if the path along the horocycle from m to p and along γ from p and p' turns left; p' is to the right of p otherwise. Note that p' is to the left of p if and only if p is to the left of p' . The shearing $s_\gamma(\Delta, \Delta')$ along γ relative to the two triangles is the signed distance between p and p' , where the sign is positive if p' is to the left of p and negative otherwise. (This is a specific case of the more general shearing defined in [Bon96, Section 2].) Note that this sign convention gives $s_\gamma(\Delta, \Delta') = s_\gamma(\Delta', \Delta)$.

Now consider the once-punctured torus $S_{1,1}$. We will define $s_\lambda: \mathcal{T}(S_{1,1}) \rightarrow \mathbf{R}^2$ when λ is the completion of a maximal-recurrent lamination on $S_{1,1}$ with finitely many leaves. Such a lamination has stump which is a simple closed curve α , and has a complement which is a pair of ideal triangles. For a given α there are two possibilities for the maximal-recurrent lamination containing it, which we denote by α_0^+ and α_0^- . Here $\alpha_0^+ = \alpha \cup \delta$ where geodesic δ spirals toward α in each direction, turning to the left as it does so. Similarly α_0^- is the union of α and a bi-infinite leaf spiraling and turning right. (Adding a leaf that spirals in opposite directions on its two ends yields a non-recurrent lamination.)

Each of these maximal-recurrent laminations of the form $\alpha \cup \delta$ has a unique completion, which is obtained by adding a pair of geodesics w, w' that emanate from the cusp and are asymptotic to one end of δ . We denote these completions by α^\pm , so $\alpha^\pm = \alpha_0^\pm \cup w \cup w'$. The laminations

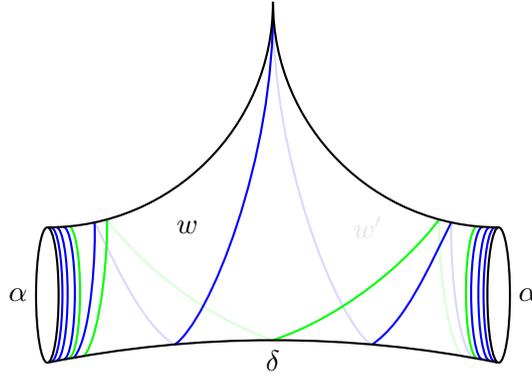


Figure 2: A simple closed geodesic α can be enlarged to a recurrent lamination $\alpha_0^+ = \alpha \cup \delta$ by adding a left-turning spiraling leaf. This lamination has a unique completion $\alpha^+ = \alpha_0^+ \cup w \cup w'$. The leaves of α^+ are shown here on the torus cut open along α .

$\alpha \subset \alpha_0^+ \subset \alpha^+$ are shown in Figure 2.

Thus we need to define the map s_λ when $\lambda = \alpha^+$ or $\lambda = \alpha^-$. Using the names for the leaves of α^\pm introduced above, we treat the two cases simultaneously. We begin with an auxiliary map $s_\lambda^0 : \mathcal{T}(S_{1,1}) \rightarrow \mathbf{R}^4$ which records a shearing parameter for each leaf of λ , and then we identify the 2-dimensional subspace of \mathbf{R}^4 that contains the image in this specific situation.

Let l be a leaf of λ and fix a lift \tilde{l} of l to $\tilde{X} = \mathbf{H}$. If l is a non-compact leaf, then l bounds two ideal triangles in X , which admit lifts Δ and Δ' with common side \tilde{l} . If $l = \alpha$ is the compact leaf, then we choose Δ and Δ' to be lifts of the two ideal triangles complementary to λ that lie on different sides of \tilde{l} and which are each asymptotic to one of the ideal points of \tilde{l} . Now define $s_l(X) = s_{\tilde{l}}(\Delta, \Delta')$, and let the $s_\lambda^0 : \mathcal{T}(S_{1,1}) \rightarrow \mathbf{R}^4$ be the map defined by

$$s_\lambda^0(X) = (s_\delta(X), s_\alpha(X), s_w(X), s_{w'}(X)).$$

We claim that in fact, $s_w(X) = s_{w'}(X) = 0$ and that $s_\delta(X) = \pm \ell_\alpha(X)$, with the sign corresponding to whether $\lambda = \alpha^+$ or $\lambda = \alpha^-$. It will then follow that s_λ^0 takes values in a 2-dimensional linear subspace of \mathbf{R}^4 , allowing us to equivalently consider the embedding $s_\lambda : \mathcal{T}(S_{1,1}) \rightarrow \mathbf{R}^2$ defined by

$$s_\lambda(X) = (\ell_\alpha(X), s_\alpha(X)).$$

To establish the claim, we consider cutting the surface X open along α to obtain a pair of pants which is further decomposed by w, w', δ into a pair of ideal triangles. The boundary lengths of this hyperbolic pair of pants are ℓ_α, ℓ_α , and 0. Gluing a pair of ideal triangles along their edges but with their edge midpoints shifted by signed distances a, b, c gives a pair of pants with boundary lengths $|a + b|, |b + c|, |a + c|$, and with the signs of $a + b, b + c, a + c$ determining the direction in which the seams spiral toward those boundary components (see [Thu86a, Section 3.9]). Specifically, a positive sum corresponds to the seam turning to the left while approaching the corresponding boundary geodesic. Applying this to our situation, for $\lambda = \alpha^+$ all spiraling leaves turn left when approaching the boundary of the pair of pants, and we obtain

$$s_w(X) + s_\delta(X) = s_{w'}(X) + s_\delta(X) = \ell_\alpha$$

and

$$s_w(X) + s_{w'}(X) = 0.$$

This gives $s_w(X) = s_{w'}(X) = 0$ and $s_\delta(X) = \ell_\alpha(X)$. For the case $\lambda = \alpha^-$ the equations are the same except with ℓ_α replaced by $-\ell_\alpha$, and the solution becomes $s_w(X) = s_{w'}(X) = 0$ and $s_\delta = -\ell_\alpha$.

Finally, we consider the effect of the various choices made in the construction of $s_\lambda(X)$. The coordinate $\ell_\alpha(X)$ is of course canonically associated to X , and independent of any choices. For $s_\alpha(X)$, however, we had to choose a pair of triangles Δ, Δ' on either side of the lift $\tilde{\alpha}$. In this case, different choices differ by finitely many moves in which one of the triangles is replaced by a neighbor on the other side of a lift of w , w' , or δ . Each such move changes the value of $s_\alpha(X)$ by adding or subtracting one of the values $s_w(X)$, $s_{w'}(X)$, or $s_\delta(X)$; this is the additivity of the shearing cocycle established in [Bon96, Section 2]. By the computation above each of these moves actually adds 0 or $\pm\ell_\alpha(X)$. Hence $s_\alpha(X)$ is uniquely determined up to addition of an integer multiple of $\ell_\alpha(X)$.

3 Twisting parameter along a Thurston geodesic

In this section, S is any surface of finite type and $\mathcal{T}(S)$ is the associated Teichmüller space.

Recall that $\mathcal{T}_{\epsilon_0}(S)$ denotes the ϵ_0 -thick part of $\mathcal{T}(S)$. Consider two points $X, Y \in \mathcal{T}_{\epsilon_0}(S)$. Recall that we say a curve α *interacts* with a geodesic lamination λ if α is a leaf of λ or if α intersects λ essentially. Suppose α is a curve that interacts with $\Lambda(X, Y)$. Let $\mathcal{G}(t)$ be any geodesic from X to Y , and let $\ell_\alpha = \min_t \ell_t(\alpha)$. We will be interested in curves which become short somewhere along $\mathcal{G}(t)$. We will call $[a, b]$ the *active interval* for α along $\mathcal{G}(t)$ if $[a, b]$ is the longest interval along $\mathcal{G}(t)$ with $\ell_\alpha(a) = \ell_\alpha(b) = \epsilon_0$. Note that any curve which is sufficiently short somewhere on $\mathcal{G}(t)$ has a nontrivial active interval.

The main goal of this section is to prove the following theorem, which in particular establishes Theorem 1.2. As in the introduction we use the notation $\text{Log}(x) = \min(1, \log(x))$.

Theorem 3.1. *Let $X, Y \in \mathcal{T}_{\epsilon_0}(S)$ and α is a curve that interacts with $\Lambda(X, Y)$. Let $\mathcal{G}(t)$ be any geodesic from X to Y and $\ell_\alpha = \min_t \ell_t(\alpha)$. Then*

$$d_\alpha(X, Y) \stackrel{*}{\asymp} \frac{1}{\ell_\alpha} \text{Log} \frac{1}{\ell_\alpha}.$$

If $\ell_\alpha < \epsilon_0$, then $d_\alpha(X, Y) \stackrel{\pm}{\asymp} d_\alpha(X_a, X_b)$, where $[a, b]$ is the active interval for α . Further, for all sufficiently small ℓ_α , the twisting $d_\alpha(X_t, \lambda)$ is uniformly bounded for all $t \leq a$ and $\ell_t(\alpha) \stackrel{}{\asymp} e^{t-b} \ell_b(\alpha)$ for all $t \geq b$.*

Note that if α is a leaf of $\Lambda(X, Y)$, then it does not have an active interval because its length grows exponentially along $\mathcal{G}(t)$, and the theorem above says that in this case there is a uniformly bounded amount of twisting along α . If α crosses a leaf of $\Lambda(X, Y)$, then $d_\alpha(X, Y)$ is large if and only if α gets short along any geodesic from X to Y . Moreover, the minimum length of α is the same for any geodesic from X to Y , up to a multiplicative constant. Further, the theorem says that all of the twisting about α occurs in the active interval $[a, b]$ of α .

Before proceeding to the proof of the theorem, we need to introduce a notion of horizontal and vertical components for a curve that crosses a leaf of $\Lambda(X, Y)$ and analyze how their lengths change in the active interval. This analysis will require some lemmas from hyperbolic geometry.

Lemma 3.2. *Let ω and ω' be two disjoint geodesics in \mathbf{H} with no endpoint in common. Let $p \in \omega$ and $p' \in \omega'$ be the endpoints of the common perpendicular between ω and ω' . Let $x \in \omega$ be arbitrary and let $x' \in \omega'$ be the point such that $d_{\mathbf{H}}(x, p) = d_{\mathbf{H}}(x', p')$ and $[x, x']$ is disjoint from $[p, p']$. Then*

$$\sinh \frac{d_{\mathbf{H}}(p, p')}{2} \cosh d_{\mathbf{H}}(x, p) = \sinh \frac{d_{\mathbf{H}}(x, x')}{2}. \quad (2)$$

For any $y \in \omega'$, we have

$$\sinh d_{\mathbf{H}}(p, p') \cosh d_{\mathbf{H}}(x, p) \leq \sinh d_{\mathbf{H}}(x, y) \quad (3)$$

and

$$d_{\mathbf{H}}(x', y) \leq d_{\mathbf{H}}(x, y). \quad (4)$$

Proof. We refer to Figure 3 for the proof. Equation (2) is well known, as the four points x, x', p', p form a Saccheri quadrilateral. The point $y' \in \omega'$ closest to x has $\angle xy'p' = \pi/2$, so x, y', p', p forms a Lambert quadrilateral and the following identity holds

$$\sinh d_{\mathbf{H}}(p, p') \cosh d_{\mathbf{H}}(x, p) = \sinh d_{\mathbf{H}}(x, y')$$

Equation (3) follows since $d_{\mathbf{H}}(x, y') \leq d_{\mathbf{H}}(x, y)$. For (4), set $A = \angle xx'y$ and $B = \angle x'xy$. We may assume A is acute; $d_{\mathbf{H}}(x, y)$ only gets bigger if A is obtuse. By the hyperbolic law of sines, we have

$$\frac{\sin A}{\sinh d_{\mathbf{H}}(x, y)} = \frac{\sin B}{\sinh d_{\mathbf{H}}(x', y)}.$$

Since $B < A < \pi/2$, we have $\sin B < \sin A$, thus $d_{\mathbf{H}}(x', y) \leq d_{\mathbf{H}}(x, y)$. \square

In this section we will often use the following elementary estimates for hyperbolic trigonometric functions. The proofs are omitted.

Lemma 3.3.

- (i) For all $0 \leq x \leq 1$, we have $\sinh(x) \stackrel{*}{\asymp} x$.
- (ii) If $0 \leq \sinh(x) \leq e^{-1}$, then $\sinh(x) \stackrel{*}{\asymp} x$.
- (iii) For all $x \geq 1$, we have $\sinh(x) \stackrel{*}{\asymp} e^x$ and $\cosh(x) \stackrel{*}{\asymp} e^x$.
- (iv) For all $x \geq 1$, we have $\operatorname{arcsinh}(x) \stackrel{\pm}{\asymp} \log(x)$ and $\operatorname{arccosh}(x) \stackrel{\pm}{\asymp} \log(x)$.

\square

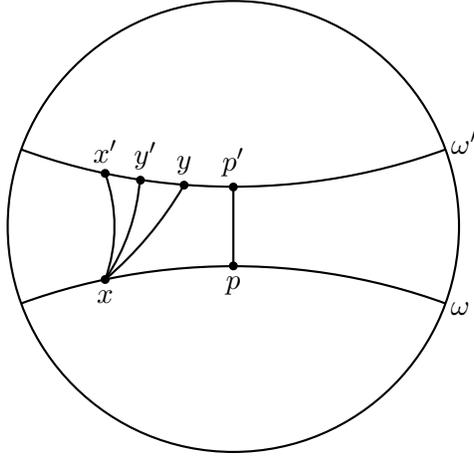


Figure 3: *Saccheri and Lambert quadrilaterals.*

Now consider $X \in \mathcal{T}(S)$ and a geodesic lamination λ on X . If α crosses a leaf ω of λ , define $V_X(\omega, \alpha)$ to be the shortest arc with endpoints on ω that, together with an arc $H_X(\omega, \alpha)$ of ω , form a curve homotopic to α . Then α passes through the midpoints of $V_X(\omega, \alpha)$ and $H_X(\omega, \alpha)$. When α is a leaf of λ , then set $H_X(\omega, \alpha) = \alpha$ and $V_X(\omega, \alpha)$ is the empty set.

Define h_X and v_X to be the lengths of $H_X(\omega, \alpha)$ and $V_X(\omega, \alpha)$ respectively. These can be computed in the universal cover $\tilde{X} \cong \mathbf{H}$ as follows. Let $\tilde{\omega}$ and $\tilde{\alpha}$ be intersecting lifts of ω and α to \mathbf{H} . Let ϕ be the hyperbolic isometry with axis $\tilde{\alpha}$ and translation length $\ell_\alpha(X)$. Set $\tilde{\omega}' = \phi(\tilde{\omega})$ and let ψ be the hyperbolic isometry taking $\tilde{\omega}$ to $\tilde{\omega}'$ with axis perpendicular to the two geodesics. Since ϕ and ψ both take ω to ω' , their composition $\psi^{-1}\phi$ is a hyperbolic isometry with axis $\tilde{\omega}$. The quantity v_X is the translation length of ψ and h_X is the translation length of $\psi^{-1}\phi$. For the latter, this means that $h_X = d_{\mathbf{H}}(\psi(q), \phi(q))$ for any $q \in \tilde{\omega}$.

In the following, let $X_t = \mathcal{G}(t)$ be a geodesic and let $\lambda = \lambda_{\mathcal{G}}$. Let α be a curve that interacts with λ . We will refer to $V_{X_t}(\omega, \alpha)$ and $H_{X_t}(\omega, \alpha)$ as the *vertical* and *horizontal* components of α at X_t . We are interested in the lengths $h_t = h_{X_t}$ and $v_t = v_{X_t}$ of the horizontal and vertical components of α as functions of t . We will show that v_t essentially decreases super-exponentially, while h_t grows exponentially. These statements are trivial if α is a leaf of λ , so we will always assume that α crosses a leaf ω of λ .

Lemma 3.4. *Suppose α crosses a leaf ω of λ . For any $t \geq s$,*

$$h_t \geq e^{t-s} (h_s - v_s).$$

Proof. In \mathbf{H} , choose a lift $\tilde{\alpha}$ of the geodesic representative of α on X_s and a lift $\tilde{\omega}$ of ω that crosses $\tilde{\alpha}$. Let $\tilde{\omega}' = \phi_s(\tilde{\omega})$ where ϕ_s is the hyperbolic isometry with axis $\tilde{\alpha}$ and translation length $\ell_s(\alpha)$. Let ψ_s be the hyperbolic isometry taking $\tilde{\omega}$ to $\tilde{\omega}'$ with axis perpendicular to the two geodesics. Let $p \in \tilde{\omega}$ be the point lying on the axis of ψ_s . By definition,

$$v_s = d_{\mathbf{H}}(p, \psi_s(p)) \quad \text{and} \quad h_s = d_{\mathbf{H}}(\psi_s(p), \phi_s(p)).$$

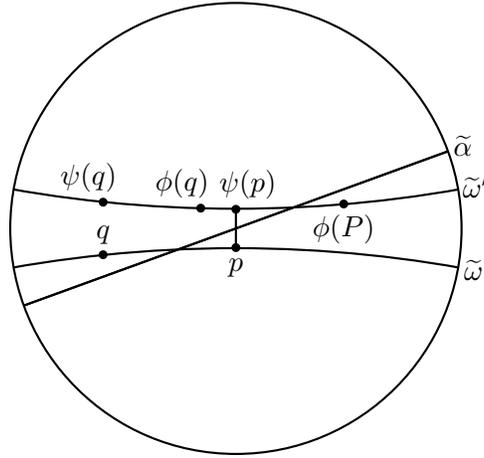


Figure 4: *Estimating h_t .*

The configuration of points and geodesics in \mathbf{H} constructed above is depicted in Figure 4; it may be helpful to refer to this figure in the calculations that follow. Note that for brevity the subscript s is omitted from the labels involving ψ, ϕ in the figure.

Let $f : X_s \rightarrow X_t$ be an optimal map and let $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ be a lift of f . Since f is an e^{t-s} -Lipschitz map such that distances along leaves of λ are stretched by a factor of exactly e^{t-s} , the images $\tilde{f}(\tilde{\omega})$ and $\tilde{f}(\tilde{\omega}')$ are geodesics and

$$d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \tilde{f}\phi_s(p)\right) = e^{t-s}h_s \quad \text{and} \quad d_{\mathbf{H}}\left(\tilde{f}(p), \tilde{f}\psi_s(p)\right) \leq e^{t-s}v_s.$$

Let ψ_t be the hyperbolic isometry taking $\tilde{f}(\tilde{\omega})$ to $\tilde{f}(\tilde{\omega}')$ with axis their common perpendicular. Let ϕ_t be the hyperbolic isometry corresponding to $f\alpha$ taking $\tilde{f}(\tilde{\omega})$ to $\tilde{f}(\tilde{\omega}')$. Note that $\phi_t\tilde{f} = \tilde{f}\phi_s$, since \tilde{f} is a lift of f . But ψ_s and ψ_t do not necessarily correspond to a conjugacy class of $\pi_1(S)$, so \tilde{f} need not conjugate ψ_s to ψ_t .

By definition,

$$h_t = d_{\mathbf{H}}\left(\psi_t\tilde{f}(p), \phi_t\tilde{f}(p)\right) = d_{\mathbf{H}}\left(\psi_t\tilde{f}(p), \tilde{f}\phi_s(p)\right).$$

By Lemma 3.2(4),

$$d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \psi_t\tilde{f}(p)\right) \leq d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \tilde{f}(p)\right).$$

Using the triangle inequality and the above equations, we obtain the conclusion.

$$\begin{aligned} h_t &\geq d_{\mathbf{H}}\left(\phi_t\tilde{f}(p), \tilde{f}\psi_s(p)\right) - d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \psi_t\tilde{f}(p)\right) \\ &\geq d_{\mathbf{H}}\left(\phi_t\tilde{f}(p), \tilde{f}\psi_s(p)\right) - d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \tilde{f}(p)\right) \\ &= d_{\mathbf{H}}\left(\tilde{f}\phi_s(p), \tilde{f}\psi_s(p)\right) - d_{\mathbf{H}}\left(\tilde{f}\psi_s(p), \tilde{f}(p)\right) \\ &\geq e^{t-s}h_s - e^{t-s}v_s \end{aligned}$$

□

Lemma 3.5. *Suppose α crosses a leaf ω of λ . There exists $\epsilon_0 > 0$ such that if $v_a \leq \epsilon_0$, then for all $t \geq a$, we have:*

$$v_t \leq e^{-Ae^{t-a}}, \quad \text{where } A \stackrel{\pm}{\asymp} \log \frac{1}{v_a}.$$

Proof. We refer to Figure 5. As before, choose a lift $\tilde{\alpha}$ to \mathbf{H} of the geodesic representative of α on X_a and a lift $\tilde{\omega}$ of ω that crosses $\tilde{\alpha}$. Let $\tilde{\omega}' = \phi(\tilde{\omega})$ where ϕ is the hyperbolic isometry with axis $\tilde{\alpha}$ and translation length $\ell(\alpha)$. Let $p \in \tilde{\omega}$ and $p' \in \tilde{\omega}'$ be the endpoints of the common perpendicular between $\tilde{\omega}$ and $\tilde{\omega}'$, so $v_a = d_{\mathbf{H}}(p, p')$.

We assume $v_a < 1$. Let $[x, y] \subset \tilde{\omega}$ and $[x', y'] \subset \tilde{\omega}'$ be segments of the same length with midpoints p and p' , such that $[x, x']$ and $[y, y']$ have length 1 and are disjoint from $[p, p']$. By (2) from Lemma 3.2,

$$d_{\mathbf{H}}(x, p) = \operatorname{arccosh} \frac{\sinh 1/2}{\sinh v_a/2}$$

We can apply Lemma 3.3(i) and (iv), which give

$$d_{\mathbf{H}}(x, y) = 2d_{\mathbf{H}}(x, p) \stackrel{\pm}{\asymp} 2 \log \frac{1}{v_a}. \quad (5)$$

In particular, v_a is small if and only if $d_{\mathbf{H}}(x, y)$ is large. We can assume v_a is small enough so that $d_{\mathbf{H}}(x, y) \geq 4$.

Let $f : X_a \rightarrow X_t$ be an optimal map and $\tilde{f} : \mathbf{H} \rightarrow \mathbf{H}$ a lift of f . Let $r \in \tilde{f}(\tilde{\omega})$ and $r' \in \tilde{f}(\tilde{\omega}')$ be the endpoints of the common perpendicular between $\tilde{f}(\tilde{\omega})$ and $\tilde{f}(\tilde{\omega}')$, so $v_t = d_{\mathbf{H}}(r, r')$. Without a loss of generality, assume that r is farther away from $\tilde{f}(x)$ than $\tilde{f}(y)$. This means

$$d_{\mathbf{H}}(\tilde{f}(x), r) \geq \frac{1}{2} d_{\mathbf{H}}(\tilde{f}(x), \tilde{f}(y)). \quad (6)$$

We also have

$$d_{\mathbf{H}}(\tilde{f}(x), \tilde{f}(y)) = e^{t-a} d_{\mathbf{H}}(x, y) \quad \text{and} \quad d_{\mathbf{H}}(\tilde{f}(x), \tilde{f}(x')) \leq e^{t-a}. \quad (7)$$

By (3) from Lemma 3.2,

$$\sinh d_{\mathbf{H}}(r, r') \cosh d_{\mathbf{H}}(\tilde{f}(x), r) \leq \sinh d_{\mathbf{H}}(\tilde{f}(x), \tilde{f}(x')).$$

Incorporating equations (6) and (7) to the above equation yields

$$\sinh d_{\mathbf{H}}(r, r') \leq \frac{\sinh e^{t-a}}{\cosh \frac{1}{2} e^{t-a} d_{\mathbf{H}}(x, y)}$$

Now use $\sinh e^{t-a} \stackrel{*}{\asymp} e^{t-a}$ and apply Lemma 3.3(iii) to the cosh term above to obtain

$$\sinh d_{\mathbf{H}}(r, r') \stackrel{*}{\asymp} e^{-e^{t-a} \left(\frac{d_{\mathbf{H}}(x, y)}{2} - 1 \right)}.$$

Finally, since $\frac{d_{\mathbf{H}}(x, y)}{2} - 1 \geq 1$, we can apply Lemma 3.3(ii) yielding $\sinh d_{\mathbf{H}}(r, r') \stackrel{*}{\asymp} d_{\mathbf{H}}(r, r')$. Setting $A = \frac{d_{\mathbf{H}}(x, y)}{2} - 1$ and applying (5) finishes the proof. \square

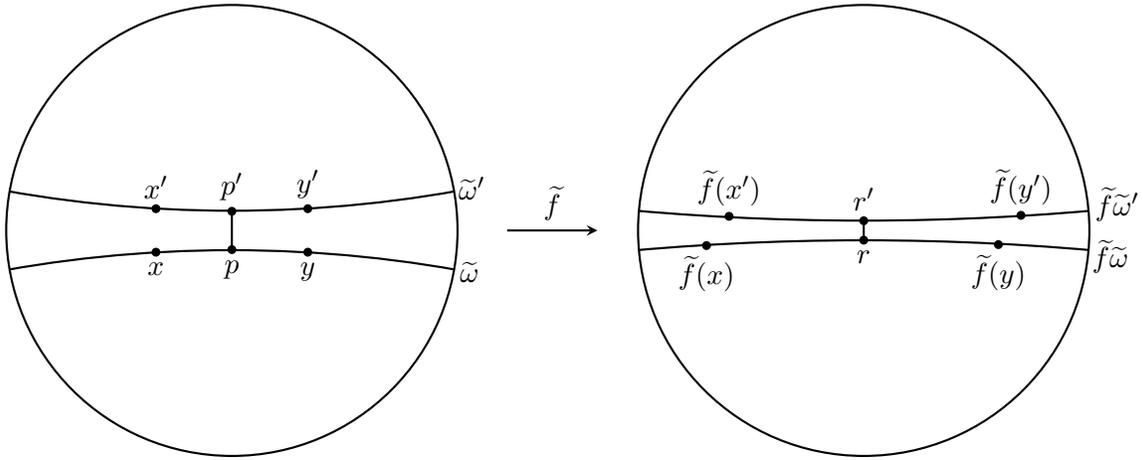


Figure 5: Bounding v_t from above.

Lemma 3.6. *Suppose α crosses a leaf ω of λ . There exists $\epsilon_1 > 0$ such that for any $\epsilon \leq \epsilon_1$, if $[a, b]$ is an interval of times with $\ell_\alpha(a) = \ell_\alpha(b) = \epsilon$, then $\ell_\alpha(t) \stackrel{*}{\gtrsim} \epsilon$ for all $t \in [a, b]$.*

Proof. We have $v_t \leq \ell_\alpha(t) \leq h_t + v_t$ and $\ell_\alpha \stackrel{*}{\gtrsim} h_t + v_t$. By Lemma 3.5, for all sufficiently small ϵ , the vertical component shrinks at least super exponentially. Since $\ell_\alpha(b) = \epsilon$, we can assume that $b - a$ is sufficiently large so that there exists $s \in [a, b]$ such that $h_s = 2v_s$. We have

$$\ell_\alpha(t) \stackrel{*}{\gtrsim} \begin{cases} v_t & \text{if } t \in [a, s], \\ h_t & \text{if } t \in [s, b]. \end{cases}$$

This means that $\ell_\alpha(t)$ decreases at least super-exponentially, reaches its minimum at time $t_\alpha \stackrel{\pm}{\approx} s$, after which it has exponential growth. In particular, this implies that α stays short on $[a, b]$. Thus, $\ell_\alpha(t) \stackrel{*}{\gtrsim} \epsilon$ for all $t \in [a, b]$. \square

Let ϵ_0 be a constant that verifies Lemma 3.5. We may assume ϵ_0 is small enough so that if $[a, b]$ is an interval of times with $\ell_\alpha(a) = \ell_\alpha(b) = \epsilon_0$, then $\ell_\alpha(t) < \epsilon_M$ for all $t \in [a, b]$, where ϵ_M is the Margulis constant. The existence of such ϵ_0 is guaranteed by Lemma 3.6. We will also assume that $\epsilon_M < 1$.

The following lemma elucidates the relationship between the relative twisting $d_\alpha(X, \lambda)$ and the length of $V_X(\omega, \alpha)$ and $\ell_\alpha(X)$.

Lemma 3.7. *Suppose α crosses a leaf ω of λ . Fix $X = X_t$ and let $\ell = \ell_\alpha(X)$ and v be the length of $V_X(\omega, \alpha)$. Then the following statements hold.*

(i) *If $\ell \leq \epsilon_M$, then*

$$d_\alpha(X, \lambda) \stackrel{\pm}{\approx} \frac{2}{\ell} \log \frac{\ell}{v}.$$

(ii) *There exists C such that if $\ell \geq \epsilon_0$ and $d_\alpha(X, \lambda) \geq C$, then*

$$v \stackrel{*}{\gtrsim} e^{-\frac{\ell}{2}(d_\alpha(X, \lambda) - 1)}.$$

Proof. The reader may find it helpful to look at Figure 4 for this proof.

Let B be the angle between $\tilde{\alpha}$ and $\tilde{\omega}$. Let L be the length of the projection of $\tilde{\omega}$ to $\tilde{\alpha}$. By definition, $d_\alpha(X, \lambda) \stackrel{\pm}{\asymp} \frac{L}{\ell}$. By hyperbolic geometry, L satisfies

$$1 = \cosh \frac{L}{2} \cdot \sin B. \quad (8)$$

To find $\sin B$, denote by ϕ the hyperbolic isometry with axis $\tilde{\alpha}$ and with translation length ℓ . Let $\tilde{\omega}' = \phi(\tilde{\omega})$. Denote by x the intersection of $\tilde{\alpha}$ and $\tilde{\omega}$ and set $x' = \phi(x)$. Let $p \in \tilde{\omega}$ and $p' \in \tilde{\omega}'$ be the points on the common perpendicular between $\tilde{\omega}$ and $\tilde{\omega}'$. That is, $p' = \psi(p)$ where ψ is the translation along an axis perpendicular to $\tilde{\omega}$ such that $\psi(\tilde{\omega}) = \tilde{\omega}'$. By construction, $d_{\mathbf{H}}(x, x') = \ell$ and $d_{\mathbf{H}}(p, p') = v$. Then the intersection point of $[p, p']$ and $[x, x']$ is the midpoint of both. Thus $\sin B$ can be found from

$$\sin B \sinh \frac{\ell}{2} = \sinh \frac{v}{2}. \quad (9)$$

Combining (8) and (9), we obtain

$$L = 2 \operatorname{arccosh} \frac{\sinh \ell/2}{\sinh v/2} \quad (10)$$

When $\ell \leq \epsilon_M < 1$, we can apply Lemma 3.3(i) and (iv) to simplify (10), obtaining:

$$L \stackrel{\pm}{\asymp} 2 \log \frac{\ell}{v} \implies d_\alpha(X, \lambda) \stackrel{\pm}{\asymp} \frac{2}{\ell} \log \frac{\ell}{v}.$$

Now let $C = \frac{2}{\epsilon_0} + 1$. If $\ell \geq \epsilon_0$ and $d_\alpha(X, \lambda) \geq C$, then $L = \ell d_\alpha(X, \lambda) \geq 1$. Thus we may apply Lemma 3.3(i) and (iii) to obtain

$$\sinh \frac{v}{2} = \frac{\sinh \ell/2}{\cosh L/2} \stackrel{*}{\asymp} e^{\frac{\ell}{2}} e^{-\frac{L}{2}} \implies \sinh \frac{v}{2} \stackrel{*}{\asymp} e^{-\frac{\ell}{2}(d_\alpha(X, \lambda) - 1)}.$$

By our assumption on C , we have $\frac{\ell}{2}(d_\alpha(X, \lambda) - 1) \geq 1$. Thus, Lemma 3.3(ii) applies and we obtain $\sinh \frac{v}{2} \stackrel{*}{\asymp} v$. This finishes the proof. \square

The following lemma implies that the length of the vertical component does not decrease too quickly.

Lemma 3.8. *Suppose α crosses a leaf ω of λ . If $\ell_\alpha(a) = \epsilon_0$ and $\ell_\alpha(t) < \epsilon_M$ for $t \geq a$, then*

$$v_t \geq e^{-Ae^{t-a}}, \quad \text{where } A \stackrel{\pm}{\asymp} \log \frac{1}{\epsilon_0}.$$

Proof. Let β be a curve that intersects α and that has smallest possible length at time a . Then $i(\alpha, \beta)$ is equal to 1 or 2. We will give the proof in the case $i(\alpha, \beta) = 1$, with the other case being essentially the same. Since α is short for all $t > a$, the part of β in a collar neighborhood of α has length that can be estimated in terms of the length of α and the relative twisting of X_t and λ (see [CRS08, Lemma 7.3]), giving a lower bound for the length of β itself:

$$\ell_\beta(t) \stackrel{\pm}{\asymp} d_\alpha(X_t, \lambda) \ell_\alpha(t) + 2 \log \frac{1}{\ell_\alpha(t)}.$$

The length of β cannot grow faster than the length of λ , therefore

$$d_\alpha(X_t, \lambda) \ell_\alpha(t) + 2 \log \frac{1}{\ell_\alpha(t)} \stackrel{\pm}{\asymp} e^{t-a} \ell_\beta(a).$$

The claim now follows from Lemma 3.7 and the fact that $\ell_\beta(a) \stackrel{\pm}{\asymp} 2 \log \frac{1}{\epsilon_0}$. \square

Theorem 3.9. *Suppose α crosses a leaf ω of λ . Let $[a, b]$ be an interval such that $\ell_\alpha(a) = \ell_\alpha(b) = \epsilon_0$. Then*

$$d_\alpha(X_a, X_b) \stackrel{*}{\asymp} \frac{1}{\epsilon_0} \log \frac{1}{\epsilon_0} \cdot e^{b-a}.$$

The length of α is minimum at a time $t_\alpha \in [a, b]$ satisfying

$$t_\alpha - a \stackrel{\pm}{\asymp} \log(b - t_\alpha).$$

And the minimum length is of order

$$e^{-Ae^{(t_\alpha - a)}} \stackrel{*}{\asymp} e^{-(b - t_\alpha)}, \text{ where } A \stackrel{\pm}{\asymp} \log \frac{1}{\epsilon_0}.$$

Proof. Note that the conclusions of the theorem are trivial if $b - a$ is small. Thus, we will require $b - a$ to be sufficiently large, as will be made more precise below.

If $h_a - v_a > \epsilon_0/2$, then by Lemma 3.4,

$$\frac{\epsilon_0}{2} e^{b-a} \leq h_b \leq \epsilon_0 \implies b - a \stackrel{\pm}{\asymp} 1.$$

Thus we may assume $h_a - v_a \leq \epsilon_0/2$. This implies that $\frac{\epsilon_0}{4} \leq v_a \leq \epsilon_0$. That is, at time $t = a$ the curve α is basically perpendicular to λ . In particular, by Lemma 3.7, $d_\alpha(X_a, \lambda) \stackrel{\pm}{\asymp} 1$.

Since $\ell_\alpha(a) = \epsilon_0$ and $\ell_\alpha(t) < \epsilon_M$ for all $t \in [a, b]$, by Lemma 3.5 and Lemma 3.8

$$v_t = e^{-Ae^{t-a}}, \text{ where } A \stackrel{\pm}{\asymp} \log \frac{1}{\epsilon_0}. \quad (11)$$

Let $s \in [a, b]$ be such that $h_s = 2v_s$; this exists when $b - a$ is sufficiently large. Lemma 3.4 gives $h_b \stackrel{*}{\asymp} v_s e^{b-s}$. Hence, since $\ell_\alpha(b) \stackrel{*}{\asymp} h_b$, we obtain

$$\epsilon_0 \stackrel{*}{\asymp} v_s e^{b-s} = e^{-Ae^{s-a}} e^{b-s} \quad (12)$$

Taking log twice we see that s satisfies

$$\log(b - s) \stackrel{\pm}{\asymp} (s - a).$$

We also see from (12) that $v_s \stackrel{*}{\asymp} e^{-(b-s)}$. This gives us the order of the minimal length of α , which is approximated by $\ell_\alpha(s) \stackrel{*}{\asymp} v_s$.

Lastly, we compute $d_\alpha(X_a, X_b)$. By Lemma 3.7 and (11),

$$\begin{aligned} d_\alpha(X_b, \lambda) &\stackrel{*}{\asymp} \frac{2}{\epsilon_0} \log \frac{\epsilon_0}{v_b} \stackrel{*}{\asymp} \frac{2}{\epsilon_0} A e^{b-a} \\ &\stackrel{*}{\asymp} \frac{2}{\epsilon_0} \log \frac{1}{\epsilon_0} e^{b-a}. \end{aligned}$$

Since $d_\alpha(X_a, \lambda) \stackrel{\pm}{\asymp} 1$, this is the desired estimate. \square

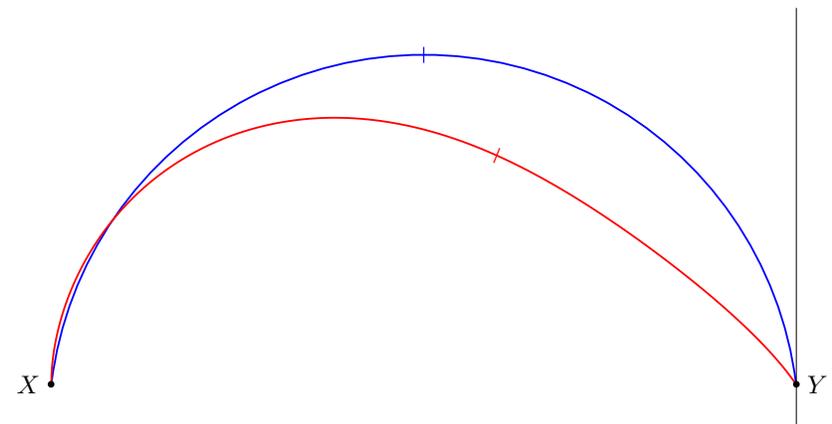


Figure 6: A Teichmüller geodesic (blue) and a stretch line (red) in the Teichmüller space $\mathcal{T}(S_{1,1}) \simeq \mathbf{H}$ of the punctured torus. Both geodesic segments start at $X = -16.302 + i$ and end at $Y = i$, and each has its midpoint marked.

Note that Theorem 3.9 highlights an interesting contrast between the behavior of Thurston metric geodesics and that of Teichmüller geodesics: Along a Teichmüller geodesic, a curve α achieves its minimum length near the midpoint of the interval $[a, b]$ in which α is short (see [Raf14, Section 3]), and this minimum is on the order of $d_\alpha(X, Y)^{-1}$. But for a Thurston metric geodesic, the minimum length occurs much closer to the start of the interval, since $t_\alpha - a$ is only on the order of $\log(b - t_\alpha)$. In addition, the minimum length on the Thurston geodesic is larger than in the Teichmüller case, though only by a logarithmic factor.

To exhibit this difference, Figure 6 shows a Teichmüller geodesic and a stretch line along β^+ joining the same pair of points in the upper half plane model of $\mathcal{T}(S_{1,1})$. Here β is a simple closed curve. In this model, the imaginary part of a point $z \in \mathbf{H}$ is approximately $\pi/\ell_\alpha(z)$, where α is a curve which has approximately the same length at both endpoints but which becomes short somewhere along each path. Thus the expected (and observed) behavior of the Thurston geodesic is that its maximum height is lower than that of the Teichmüller geodesic, but that this maximum height occurs closer to the starting point for the Thurston geodesic. Further properties of Thurston geodesics in the punctured torus case are explored in the next section.

Continuing toward the proof of Theorem 3.1, we show:

Lemma 3.10. *Suppose α crosses a leaf ω of λ . There exists $C > 0$ such that if $\ell_\alpha(s) \geq \epsilon_0$ and $d_\alpha(X_s, \lambda) \geq C$, then $\ell_\alpha(t) \stackrel{*}{\asymp} e^{t-s} \ell_\alpha(s)$ for all $t \geq s$.*

Proof. By Lemma 3.7, if $d_\alpha(X_s, \lambda)$ is sufficiently large, then

$$v_s \stackrel{*}{\asymp} e^{-\frac{\ell}{2}(d_\alpha(X_s, \lambda) - 1)}.$$

In particular, for sufficiently large $d_\alpha(X_s, \lambda)$, we can guarantee that $h_s - v_s \stackrel{*}{\asymp} h_s$ and so $\ell_\alpha(s) \stackrel{*}{\asymp} h_s$. By Lemma 3.4, for all $t > s$,

$$\ell_\alpha(t) \geq h_t \stackrel{*}{\asymp} e^{t-s} h_s \stackrel{*}{\asymp} e^{t-s} \ell_\alpha(s).$$

On the other hand, $\ell_\alpha(t) \leq e^{t-s}\ell_\alpha(s)$. This finishes the proof. \square

Lemma 3.11. *Suppose α interacts with λ . If $\ell_\alpha(t) \geq \epsilon_0$ for all $t \in [a, b]$, then $d_\alpha(X_a, X_b) \stackrel{\pm}{\asymp} 1$.*

Proof. We first show that for any $s \leq t$, if $\ell_\alpha(t) \stackrel{*}{\asymp} e^{t-s}\ell_\alpha(s)$, then $d_\alpha(X_s, X_t) \stackrel{\pm}{\asymp} 1$. Let β be the shortest curve at X_s that intersects α . At time t , the length of β satisfies

$$\frac{\ell_\beta(t)}{\ell_\alpha(t)} \stackrel{\pm}{\asymp} d_\alpha(X_s, X_t).$$

Therefore, since $\ell_\beta(t) \leq e^{t-s}\ell_\beta(s)$ and $\ell_\alpha(b) \stackrel{*}{\asymp} e^{t-s}\ell_\alpha(s)$, we have

$$d_\alpha(X_s, X_t) \stackrel{\pm}{\asymp} \frac{\ell_\beta(s)}{\ell_\alpha(s)} \stackrel{*}{\asymp} \frac{2}{\epsilon_0} \log \frac{1}{\epsilon_0} \stackrel{\pm}{\asymp} 1.$$

If α is a leaf of λ , then $\ell_\alpha(b) = e^{b-a}\ell_\alpha(a)$, so the conclusion follows from the paragraph above. Now suppose that α crosses a leaf of λ . Let C be the constant of Lemma 3.10. If $d_\alpha(X_t, \lambda) < C$ for all $t \in [a, b]$, then we are done. Otherwise, there is an earliest time $t \in [a, b]$ such that $d_\alpha(X_t, \lambda) \geq C$. It is immediate that $d_\alpha(X, X_t) \stackrel{\pm}{\asymp} 1$. By Lemma 3.10, $\ell_\alpha(b) \stackrel{*}{\asymp} e^{b-t}\ell_\alpha(t)$, so $d_\alpha(X_t, X_b) \stackrel{\pm}{\asymp} 1$ by the above paragraph. The result follows. \square

We will now prove the theorem stated at the beginning of this section.

Proof of Theorem 3.1. If $\ell_\alpha \geq \epsilon_0$, then by Lemma 3.11.

$$d_\alpha(X, Y) \stackrel{\pm}{\asymp} 1 \stackrel{\pm}{\asymp} \frac{1}{\ell_\alpha} \text{Log} \frac{1}{\ell_\alpha}.$$

Now suppose $\ell_\alpha < \epsilon_0$ and let $[a, b]$ be active interval for α . From Theorem 3.9, the minimal length ℓ_α occurs at $t_\alpha \in [a, b]$ satisfying $t_\alpha - a \stackrel{\pm}{\asymp} \log(b - t_\alpha)$. We have

$$d_\alpha(X_a, X_b) \stackrel{*}{\asymp} e^{b-a} \stackrel{*}{\asymp} e^{(b-t_\alpha)} (b - t_\alpha) \stackrel{*}{\asymp} \frac{1}{\ell_\alpha} \log \frac{1}{\ell_\alpha}.$$

Since $\ell_\alpha < \epsilon_0$, we have $\frac{1}{\ell_\alpha} \log \frac{1}{\ell_\alpha} \stackrel{\pm}{\asymp} \frac{1}{\ell_\alpha} \text{Log} \frac{1}{\ell_\alpha}$, with equality for ℓ_α small enough. By Lemma 3.11, $d_\alpha(X, X_a)$ and $d_\alpha(X_b, Y)$ are both uniformly bounded. Thus $d_\alpha(X, Y) \stackrel{\pm}{\asymp} d_\alpha(X_a, X_b)$.

For the last statement, let C be the constant of Lemma 3.10. By assumption $\ell_\alpha(t) > \epsilon_0$ for all $t \leq a$. If there exists $t \leq a$ such that $d_\alpha(X_t, \lambda) \geq C$, then $\ell_\alpha(t_\alpha) \stackrel{*}{\asymp} e^{t_\alpha-s}\ell_s(\alpha)$, where t_α is the time of the minimal length of ℓ_α . This is impossible for all sufficiently small ℓ_α . Finally, since $d_\alpha(X_a, X_b) \stackrel{*}{\asymp} \frac{1}{\ell_\alpha} \log \frac{1}{\ell_\alpha}$, for all sufficiently small ℓ_α , we can guarantee that $d_\alpha(X_b, \lambda) \geq C$. The final conclusion follows by Lemma 3.10. \square

Recall that two curves that intersect cannot both have lengths less than ϵ_M at the same time. Therefore, if α and β intersect and $\ell_\alpha < \epsilon_0$ and $\ell_\beta < \epsilon_0$, then their active intervals must be disjoint. This defines an ordering of α and β along \mathcal{G} . In the next section, we will focus on the torus $S_{1,1}$ and show that the order of α and β along \mathcal{G} will always agree with their order in the projection of $\mathcal{G}(t)$ to the Farey graph.

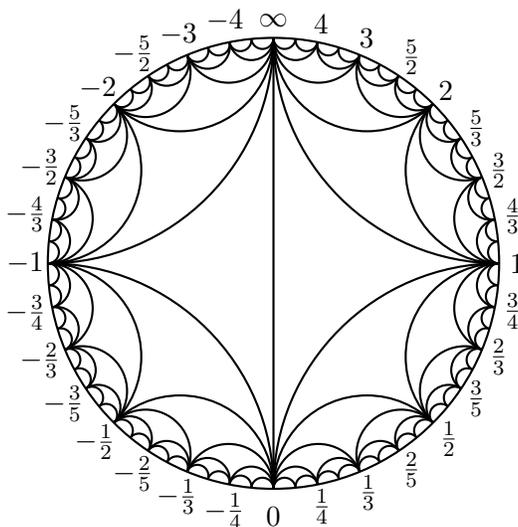


Figure 7: *The Farey graph.*

4 Coarse description of geodesics in $\mathcal{T}(S_{1,1})$

4.1 Farey graph

See [Min99] for background on the Farey graph.

Let $S_{1,1}$ be the once-punctured torus and represent its universal cover by the upper half plane \mathbf{H} , where we identify $\partial\mathbf{H} = \mathbf{R} \cup \{\infty\}$. The point ∞ is considered an extended rational number with reduced form $1/0$. Fix a basis $\{\alpha_1, \alpha_2\}$ for $H_1(S_{1,1})$, and associate to any simple closed curve α on S the rational number $p/q \in \partial\mathbf{H}$, where $\alpha = -p\alpha_1 + q\alpha_2$. Given two curves $\alpha = p/q$ and $\beta = r/s$ in reduced fractions, their geometric intersection number is given by the formula $|ps - rq|$. Connect α and β by a hyperbolic geodesic if $|ps - rq| = 1$. The resulting graph \mathcal{F} is called the Farey graph, which is homeomorphic to the curve graph of $S_{1,1}$.

Fix a curve $\alpha \in \mathcal{F}$. Let $\beta_0 \in \mathcal{F}$ be any curve with $i(\alpha, \beta_0) = 1$. The Dehn twist family about α is $\beta_n = D_\alpha^n(\beta_0)$. This is exactly the set of vertices of \mathcal{F} that are connected to α by an edge. We will use the convention that positive Dehn twists increase slope. With this convention, the Dehn twist family of any curve will induce the counter-clockwise orientation on $\partial\mathbf{H}$.

4.2 Markings and pivots

A *marking* on $S_{1,1}$ is an unordered pair of curves $\{\alpha, \beta\}$ such that $i(\alpha, \beta) = 1$. Given a marking $\{\alpha, \beta\}$, there are four markings that are obtained from μ by an *elementary move*, namely:

$$\{\alpha, D_\alpha(\beta)\}, \quad \{\alpha, D_\alpha^{-1}(\beta)\}, \quad \{\beta, D_\beta(\alpha)\}, \quad \{\beta, D_\beta^{-1}(\alpha)\}.$$

Denote by \mathcal{MG} the graph with markings as vertices and an edge connecting two markings that differ by an elementary move. Then \mathcal{MG} has the following property.

Lemma 4.1. *For any $\mu, \mu' \in \mathcal{MG}$, there exists a unique geodesic connecting them.*

Proof. The set of markings on $S_{1,1}$ can be identified with the set of edges of \mathcal{F} . Each edge of \mathcal{F} separates \mathbf{H} into two disjoint half-spaces. Let $E(\mu, \mu')$ be the set of edges in \mathcal{F} that separate the interior of μ from the interior of μ' . Set $\overline{E}(\mu, \mu') = E(\mu, \mu') \cup \{\mu, \mu'\}$. Every $\nu \in \overline{E}(\mu, \mu')$ must appear in every geodesic from μ and μ' and any $\nu \in \mathcal{MG}$ lying on a geodesic from μ to μ' must lie in $\overline{E}(\mu, \mu')$. For each $\nu \in \overline{E}(\mu, \mu')$, let H_ν be the half-space in \mathbf{H} containing the interior of μ' . There is a linear order on $\overline{E}(\mu, \mu') = \{\mu_1 < \mu_2 < \dots < \mu_n\}$ induced by the relation $\mu_i < \mu_{i+1}$ if and only if $H_{\mu_i} \supset H_{\mu_{i+1}}$. The sequence $\mu = \mu_1, \mu_2, \dots, \mu_n = \mu'$ is the unique geodesic path in \mathcal{MG} from μ to μ' . \square

Given two markings μ and μ' and a curve α , let n_α be the number of edges in $\overline{E}(\mu, \mu')$ containing α . We say α a *pivot* for μ and μ' if $n_\alpha \geq 2$, and n_α is the *coefficient* of the pivot. Let $\text{Pivot}(\mu, \mu')$ be the set of pivots for μ and μ' . This set is naturally linearly ordered as follows. Given $\alpha \in \text{Pivot}(\mu, \mu')$, let e_α be the last edge in $\overline{E}(\mu, \mu')$ containing α . Then for $\alpha, \beta \in \text{Pivot}(\mu, \mu')$, we set $\alpha < \beta$ if e_α appears before e_β in g .

Recall that in Section 2.3 we defined the unsigned twisting (along α) for a pair of curves β, β' ; this is denoted $d_\alpha(\beta, \beta')$. Generalizing this, we define unsigned twisting for the pair of markings μ, μ' by

$$d_\alpha(\mu, \mu') = \min_{\beta \subset \mu, \beta' \subset \mu'} d_\alpha(\beta, \beta'),$$

where β is a curve in μ and β' is a curve in μ' . Similarly we define $d_\alpha(\beta, \mu') = \min_{\beta' \subset \mu'} d_\alpha(\beta, \beta')$. In terms of these definitions, we have:

Lemma 4.2 ([Min99]). *For any $\mu, \mu' \in \mathcal{MG}$ and curve α , we have $n_\alpha \stackrel{\pm}{\geq} d_\alpha(\mu, \mu')$. For $\alpha, \beta \in \text{Pivot}(\mu, \mu')$, if $\alpha < \beta$, then $d_\alpha(\beta, \mu') \stackrel{\pm}{\geq} 1$ and $d_\beta(\mu, \alpha) \stackrel{\pm}{\geq} 1$. Conversely, if n_α is sufficiently large and $d_\alpha(\beta, \mu') \stackrel{\pm}{\geq} 1$, then $\alpha < \beta$.*

Let ϵ_0 be the constant of the previous section. For any $X \in \mathcal{T}_{\epsilon_0}(S_{1,1})$, we will define a *short marking* on X as follows. Note that on a torus, the set A of systoles on X has cardinality at most three and any pair in A intersects exactly once. If A has cardinality at least 2, then a short marking on X is any choice of a pair of curves in A . If X has a unique systole, then the set A' of the shortest and the second shortest curves on X again has cardinality at most three and every pair in A' intersects exactly once. A short marking on X is any choice of pair of curves in A' . Note that in our definition, there are either a uniquely defined short marking or three short markings on X . In the latter case, the three short markings on X form a triangle in \mathcal{F} . This implies that, given $X, Y \in \mathcal{T}_{\epsilon_0}(S_{1,1})$, there are well-defined short markings μ_X and μ_Y on X and Y such that $d_{\mathcal{MG}}(\mu_X, \mu_Y)$ is minimal among all short markings on X and Y . By Lemma 4.1, the geodesic g from μ_X to μ_Y is unique. We will denote by $\text{Pivot}(X, Y) = \text{Pivot}(\mu_X, \mu_Y)$ and refer to $\text{Pivot}(X, Y)$ as the set of pivots for X and Y .

Given $X, Y \in \mathcal{T}(S_{1,1})$, we have that $d_\alpha(X, Y) \stackrel{\pm}{\geq} d_\alpha(\mu_X, \mu_Y)$.

The following statements establish Theorem 1.3 of the introduction.

Theorem 4.3. *Suppose $X, Y \in \mathcal{T}_{\epsilon_0}(S_{1,1})$ and let $\mathcal{G}(t)$ be any geodesic from X to Y . Let $\ell_\alpha = \inf_t \ell_\alpha(t)$. There are positive constants $\epsilon_1 \leq \epsilon_0$, C_1 , and C_2 such that*

- (i) *If $\ell_\alpha \leq \epsilon_1$, then $\alpha \in \text{Pivot}(X, Y)$ and $d_\alpha(X, Y) \geq C_1$.*
- (ii) *If $d_\alpha(X, Y) \geq C_2$, then $\ell_\alpha \leq \epsilon_1$ and $\alpha \in \text{Pivot}(X, Y)$.*
- (iii) *Suppose α and β are curves such that there exist $s, t \in I$ with $\ell_\alpha(s) \leq \epsilon_1$ and $\ell_\alpha(t) \leq \epsilon_1$. Then $\alpha < \beta$ in $\text{Pivot}(X, Y)$ if and only if $s < t$.*
- (iv) *For any $\alpha \in \text{Pivot}(X, Y)$, $\ell_\alpha \stackrel{*}{\asymp} 1$.*

Proof. Let $\lambda = \Lambda(X, Y)$. On the torus, every curve α intersects with λ . If λ contains α , then $\ell_\alpha(t) = e^t \ell_\alpha(X)$. But this implies $\ell_\alpha = \ell_X(\alpha) \geq \epsilon_0$ and $d_\alpha(X, Y) \stackrel{\pm}{\asymp} 1$ by Lemma 3.10. Thus, we may assume that α crosses a leaf of λ . By Theorem 3.1, $d_\alpha(X, Y) \stackrel{*}{\asymp} \frac{1}{\ell_\alpha} \log \frac{1}{\ell_\alpha}$. This establishes the existence of $\epsilon_1 \leq \epsilon_0$, C_1 , and C_2 , for (i) and (ii).

For (iii), since $\ell_\alpha < \epsilon_1$ and $\ell_\beta < \epsilon_1$, they are both pivots by statement (i). Let $[a, b]$ be the active interval for α . Recall that this means that $\ell_\alpha(a) = \ell_\beta(b) = \epsilon_0$ and $\ell_\alpha(t) < \epsilon_M$ for all $t \in [a, b]$, where ϵ_M is the Margulis constant. Similarly, let $[c, d]$ be the active interval for β . On the torus two curves always intersect, so α and β cannot simultaneously be shorter than ϵ_M , so $[a, b]$ and $[c, d]$ must be disjoint. By Lemma 4.2 we have $\alpha < \beta$ if and only if $d_\alpha(\beta, \mu_Y) \stackrel{\pm}{\asymp} d_\alpha(X_c, Y) \stackrel{\pm}{\asymp} 1$, which holds if and only if $b < c$ by Theorem 3.1.

For (iv), let $\alpha \in \text{Pivot}(X, Y)$ and assume $\ell_\alpha > \epsilon_1$. Let $e \in \overline{E}(X, Y)$ be an edge containing α . Let β be the other curve of e . The edge e separates X and Y , so any geodesic $\mathcal{G}(t)$ from X to Y must cross e at some point X_t . If $\ell_\beta(t) > \epsilon_1$, then neither α or β is short on X_t , so X_t lies near the midpoint of e . This implies that $\ell_\alpha(t) \stackrel{*}{\asymp} 1$. On the other hand, if $\ell_\beta(t) \leq \epsilon_1$, then β is a pivot by (i). Either $\alpha < \beta$ or $\beta < \alpha$ in $\text{Pivot}(X, Y)$. If $\alpha < \beta$, then $d_\beta(X, \alpha) \stackrel{\pm}{\asymp} 1$ by Lemma 4.2. Let $[a, b]$ be the active interval for β . By Theorem 3.1 we have $d_\beta(X, X_a) \stackrel{\pm}{\asymp} 1$, and d_β satisfies the triangle inequality up to additive error (by [MM00, Equation 2.5]), so we conclude $d_\beta(\alpha, X_a) \stackrel{\pm}{\asymp} 1$. This, together with $\ell_\beta(a) = \epsilon_0$, yields $\ell_\alpha(a) \stackrel{*}{\asymp} 1$. If $\beta < \alpha$, then the same argument using X_b and Y in place of X_a and X also yields $\ell_\alpha(b) \stackrel{*}{\asymp} 1$. This concludes the proof. \square

5 Envelopes in $\mathcal{T}(S_{1,1})$

5.1 Fenchel-Nielsen coordinates along stretch paths in $\mathcal{T}(S_{1,1})$

We now focus on the once-punctured torus $S_{1,1}$, and on the completions α^\pm of maximal-recurrent laminations containing a simple closed curve α discussed in Section 2.5.

Consider the curve α as a pants decomposition of S and define $\tau_\alpha(X)$ to be the Fenchel-Nielsen twist coordinate of X relative to α . Note that $\tau_\alpha(X)$ is well defined up to a multiple of $\ell_\alpha(X)$, and after making a choice at some point, $\tau_\alpha(X)$ is well defined. The Fenchel-Nielsen theorem states that the pair of functions $(\log \ell_\alpha(\cdot), \tau_\alpha(\cdot))$ define a diffeomorphism of $\mathcal{T}(S_{1,1}) \rightarrow \mathbf{R}^2$.

Each α^\pm defines a foliation \mathcal{F}_α^\pm on $\mathcal{T}(S_{1,1})$ whose leaves are the α^\pm -stretch lines. In the α^+ shearing coordinate system, this is the foliation of the convex cone in \mathbf{R}^2 that is the image of $\mathcal{T}(S_{1,1})$ by open rays from the origin.

In this section we denote a point on the α^\pm stretch line through X by $X_t^\pm = \text{stretch}(X, \alpha^\pm, t)$. The function $\log \ell_\alpha(X_t^\pm) = \log \ell_\alpha(X) + t$ is smooth in t . Our first goal is to establish the following theorem.

Theorem 5.1. *For any simple closed curve α on $S_{1,1}$ and any point $X = X_0 \in \mathcal{T}(S_{1,1})$, the functions $\tau_\alpha(X_t^\pm)$ are smooth in t . Further,*

$$\frac{d}{dt}\tau_\alpha(X_t^+)|_{t=0} > \frac{d}{dt}\tau_\alpha(X_t^-)|_{t=0}.$$

That is, the pair of foliations \mathcal{F}_α^+ and \mathcal{F}_α^- are smooth and transverse.

We proceed to prove smoothness of $\tau_\alpha(X_t^+)$. Recall that the α^+ shearing embedding is $s_{\alpha^+}(X) = (\ell_\alpha(X), s_\alpha(X))$ where $s_\alpha(X)$ is defined in Section 2.5, and that like τ_α , the function s_α is defined only up to adding an integer multiple of $\ell_\alpha(X)$. To further lighten our notation, we will often write $\ell_\alpha(t)$ instead of $\ell_\alpha(X_t^+)$, and $s_\alpha(t)$ for $s_\alpha(X_t^+)$.

We also denote $\tau_\alpha(0)$ and $\ell_\alpha(0)$ by τ_0 and ℓ_0 respectively. Note that the values of τ_α and ℓ_α do not depend on the choice of α^+ or α^- but the values of the shearing coordinates do.

We know, from the description of stretch paths in Section 2.4, that

$$s_\alpha(t) = s_\alpha(0)e^t \quad \text{and} \quad \ell_\alpha(t) = \ell_0 e^t.$$

We can now compute $\tau_\alpha(X_t^+)$ as follows, referring to Figure 8. Fix a lift $\tilde{\alpha}$ of α to be the imaginary line (shown in red in Figure 8) in the upper half-plane \mathbf{H} . Now develop the picture on both sides of $\tilde{\alpha}$. Since we are considering α^+ , all the triangles on the left of $\tilde{\alpha}$ are asymptotic to ∞ and all the triangles on the right of $\tilde{\alpha}$ are asymptotic to 0. Below we will choose some normalization, but first note there is an orientation-preserving involution of $S_{1,1}$, the elliptic involution, that exchanges T and T' while preserving α as a set. Let $i: \mathbf{H} \rightarrow \mathbf{H}$ be a lift of this involution chosen to preserve $\tilde{\alpha}$, which therefore has the form $i(z) = e^c/z$ for some $c = c_t \in \mathbf{R}$. Notice that i exchanges the two sides of $\tilde{\alpha}$ and that it fixes a unique point $e^{c/2}$ in \mathbf{H} .

To fix the shearing coordinate $s_\alpha(t)$, we make the choice of triangles in \mathbf{H} required by the construction of Section 2.5. Choose two triangles Δ_l and Δ_r in \mathbf{H} separated by $\tilde{\alpha}$ so that one is a lift of T and the other is a lift of T' and $i(\Delta_r) = \Delta_l$. Let \tilde{w} be the edge of Δ_l that is a lift of w , namely, $\tilde{w} = [x, \infty]$ for some $x < 0$. Let $\phi_\alpha(z) = e^{\ell_\alpha} z$ be the isometry associated to $\tilde{\alpha}$ oriented toward ∞ . The image $\phi_\alpha(\tilde{w}) = [e^{\ell_\alpha} x, \infty]$ is another lift of w . Let $\tilde{\delta}$ be the lift of δ that is asymptotic to \tilde{w} and $\phi_\alpha(\tilde{w})$. By applying a further dilation to the picture if necessary, we can assume that $\tilde{\delta} = [x-1, \infty]$. Now, the geodesic $\tilde{w}' = [x, x-1]$ is a lift of w' .

With our normalization, the midpoint of $[x, \infty]$ associated to Δ_l is the point $(x, 1)$. Since $|s_\delta| = |\ell_\alpha|$, we have the relative shearing between triangles $\Delta_l = [x, x-1, \infty]$ and $[x-1, e^{\ell_\alpha}, \infty]$

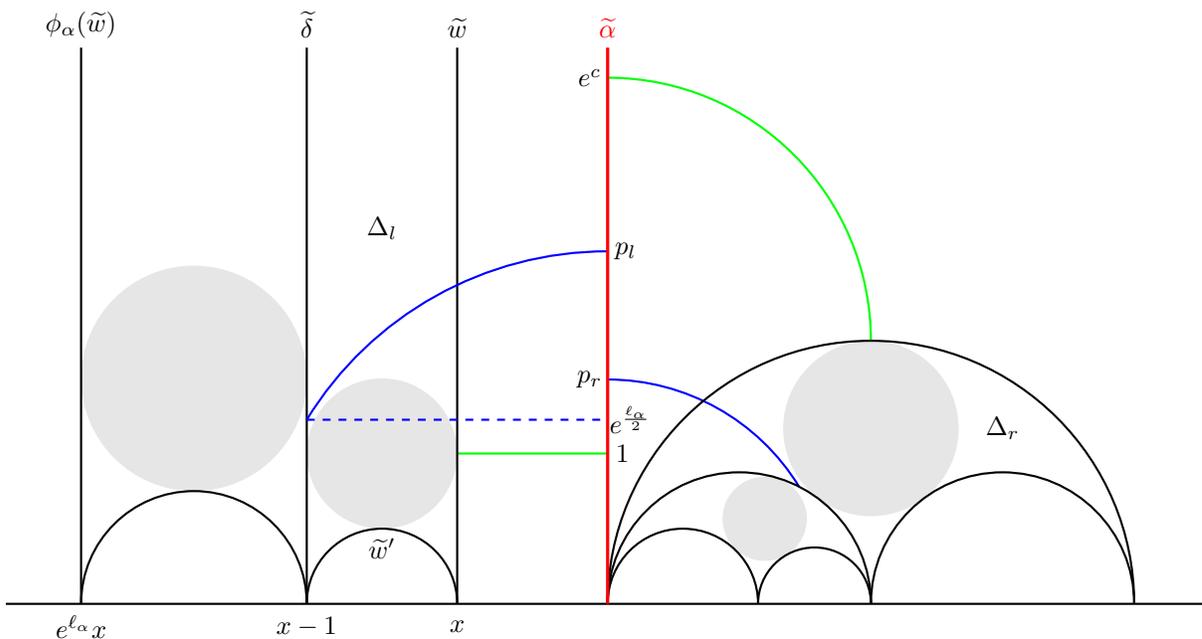


Figure 8: Computing the Fenchel-Nielsen twist along the α^+ stretch path.

is e^{ℓ_α} . That is,

$$\frac{|e^{\ell_\alpha}x - (x-1)|}{|(x-1) - x|} = e^{\ell_\alpha}$$

from which it follows that

$$x = -\coth(\ell_\alpha/2).$$

Let h_l be the endpoint on $\tilde{\alpha}$ of the horocycle based at infinity containing the midpoint of \tilde{w} considered as an edge of Δ_l . Let $h_r = i(h_l)$. By construction, $h_l = 1$ and $h_r = e^c$. We can normalize so that $s_\alpha = c$.

To visualize the twisting $\tau_+(t)$ at X_t about α , consider the geodesic arc β with both endpoints on α intersecting perpendicularly. By symmetry, this arc intersects δ at a point q that is equidistant to the midpoints of δ associated to T and T' . We choose a lift $\tilde{\beta}$ that passes through $\tilde{q} = (x-1, e^{\ell_\alpha/2})$. Let p_l be the endpoint on $\tilde{\alpha}$ of the lift of β that passes through $\tilde{\delta}$. Since $\tilde{\beta}$ is perpendicular to $\tilde{\alpha}$, we have q and p_l lie on a Euclidean circle centered at the origin. Using the Pythagorean theorem, we obtain:

$$p_l = \sqrt{(x-1)^2 + (e^{\ell_\alpha/2})^2} = e^{\ell_\alpha/2} \coth \frac{\ell_\alpha}{2}.$$

Let $p_r = i(p_l) = e^c/p_l$. We can normalize so that the twisting $\tau_\alpha(t)$ is the signed distance

between p_l and p_r . That is

$$\begin{aligned}\tau_\alpha &= \log \frac{p_r}{p_l} = \log \frac{e^c}{e^{\ell_\alpha} \coth^2(\frac{\ell_\alpha}{2})} \\ &= c - \ell_\alpha - 2 \log \coth \frac{\ell_\alpha}{2} \\ &= s_\alpha - \ell_\alpha - 2 \log \coth \frac{\ell_\alpha}{2}\end{aligned}\tag{13}$$

As we mentioned previously, $s_\alpha(t) = s_\alpha(0)e^t$ and $\ell_\alpha(t) = \ell_0 e^t$. Hence,

$$\tau_\alpha(t) = e^t(s_\alpha(0) - \ell_0) - 2 \log \coth \frac{e^t \ell_0}{2}.$$

However, since we have

$$s_\alpha(0) - \ell_0 = \tau_0 + 2 \log \coth \frac{\ell_0}{2}$$

we obtain

$$\tau_\alpha(X_t^+) = e^t \tau_0 + 2e^t \log \coth \frac{\ell_0}{2} - 2 \log \coth \frac{e^t \ell_0}{2}.\tag{14}$$

Now let X_t^- be the stretch path starting from X associated to α^- . By a similar computation, we can show that the twisting on this path is given by

$$\tau_\alpha(X_t^-) = e^t \tau_0 - 2e^t \log \coth \frac{\ell_0}{2} + 2 \log \coth \frac{e^t \ell_0}{2}.\tag{15}$$

This shows that $\tau(X_t^+)$ and $\tau(X_t^-)$ are both smooth functions of t . Further, by a simple computation, we see that

$$\frac{d}{dt} \left(\tau_\alpha(X_t^+) - \tau_\alpha(X_t^-) \right) \Big|_{t=0} = 4 \log \coth \frac{\ell_0}{2} + 2\ell_0 \tanh \frac{\ell_0}{2} \operatorname{csch}^2 \frac{\ell_0}{2} > 0.\tag{16}$$

This finishes the proof of Theorem 5.1.

5.2 Structure of envelopes

Our goal in this section is to prove Theorem 1.1 of the introduction. This is divided into several propositions within this section.

For any surface S of finite type and a recurrent lamination λ on S and $X \in \mathcal{T}(S)$, define

$$\operatorname{Out}(X, \lambda) = \{Z \in \mathcal{T}(S) : \lambda = \Lambda(X, Z)\} \quad \text{and} \quad \operatorname{In}(X, \lambda) = \{Z \in \mathcal{T}(S) : \lambda = \Lambda(Z, X)\}.$$

We call these the *out-envelope* and *in-envelope* of X (respectively) in the direction λ .

Proposition 5.2. *The out-envelopes and in-envelopes have the following properties.*

- (i) *If λ is maximal-recurrent, then $\operatorname{Out}(X, \lambda)$ is the unique stretch ray along the completion of λ starting at X , and $\operatorname{In}(X, \lambda)$ is the unique stretch ray along the completion of λ terminating at X .*

(ii) The closure of $\text{Out}(X, \lambda)$ consists of points Y with $\lambda \subset \Lambda(X, Y)$. Similarly, the closure of $\text{In}(X, \lambda)$ is a set of points Y with $\lambda \subset \Lambda(X, Y)$.

(iii) If λ is a simple closed curve, then $\text{Out}(X, \lambda)$ and $\text{In}(X, \lambda)$ are open sets.

Proof. First assume λ is maximal-recurrent and let $\widehat{\lambda}$ be its completion. Thurston showed in [Thu86b, Theorem 8.5] that if $\Lambda(X, Y) = \lambda$, then there exists $t > 0$ such that $Y = \text{stretch}(X, \widehat{\lambda}, t)$. That is, any point in $\text{Out}(X, \lambda)$ can be reached from X by the forward stretch ray along $\widehat{\lambda}$ and any point in $\text{In}(X, \lambda)$ can be reached from X by the backward stretch ray along $\widehat{\lambda}$. This is (i).

For the other statements, we use [Thu86b, Theorem 8.4], which shows that if Y_i converges Y , then any limit point of $\Lambda(X, Y_i)$ in the Hausdorff topology is contained in $\Lambda(X, Y)$. Applying this to a point Y in the closure of $\text{Out}(X, \lambda)$, and a sequence $Y_i \in \text{Out}(X, \lambda)$ converging to Y , we obtain (ii) for $\text{Out}(X, \lambda)$. The analogous statement for $\text{In}(X, \lambda)$ is proven similarly. To obtain (iii), let λ be a simple closed curve, $Y \in \text{Out}(X, \lambda)$, and Y_i is any sequence converging to Y , then any limit point of $\Lambda(X, Y_i)$ is contained in λ . Since λ is a simple closed curve, $\Lambda(X, Y_i) = \lambda$ for all sufficiently large i . This shows $\text{Out}(X, \lambda)$ is open. The same proof also applies to $\text{In}(X, \lambda)$. \square

Let $X, Y \in \mathcal{J}(S)$, and denote $\lambda = \Lambda(X, Y)$. We define the envelope of geodesics from X to Y to be the set

$$\text{Env}(X, Y) = \left\{ Z : Z \in [X, Y] \text{ for some geodesic } [X, Y] \right\}.$$

Proposition 5.3. For any $X, Y \in \mathcal{J}(S)$, $\text{Env}(X, Y) = \overline{\text{Out}(X, \lambda)} \cap \overline{\text{In}(Y, \lambda)}$.

Proof. For any $Z \in \text{Env}(X, Y)$, since Z lies on a geodesic from X to Y , λ must be contained in $\Lambda(X, Z)$ and in $\Lambda(Z, Y)$. This shows $\text{Env}(X, Y) \subset \overline{\text{Out}(X, \lambda)} \cap \overline{\text{In}(Y, \lambda)}$. On the other hand, if $Z \in \overline{\text{Out}(X, \lambda)} \cap \overline{\text{In}(Y, \lambda)}$, then $\lambda \subset \Lambda(X, Z)$ and $\lambda \subset \Lambda(Z, Y)$. That is, if μ is the stump of λ , then $d_{\text{Th}}(X, Z) = \log \frac{\ell_{\mu}(Z)}{\ell_{\mu}(X)}$ and $d_{\text{Th}}(Z, Y) = \log \frac{\ell_{\mu}(Y)}{\ell_{\mu}(Z)}$, so $d_{\text{Th}}(X, Y) = d_{\text{Th}}(X, Z) + d_{\text{Th}}(Z, Y)$. Thus, the concatenation of any geodesic from X to Z and from Z to Y is a geodesic from X to Y . \square

Now we focus on the structure of envelopes in the case that $S = S_{1,1}$.

Proposition 5.4. Let α be a simple closed curve on $S_{1,1}$. For any $X \in \mathcal{J}(S_{1,1})$, the set $\text{Out}(X, \alpha)$ is an open region bounded by the two stretch rays along α^{\pm} emanating from X . Similarly, $\text{In}(X, \alpha)$ is an open region bounded by the two stretch rays along α^{\pm} terminating at X .

Proof. Set $X_t^{\pm} = \text{stretch}(X, \alpha^{\pm}, t)$. For any surface S and every two points $X, Y \in \mathcal{J}(S)$, Thurston constructed a geodesic from X to Y that is a concatenation of stretch paths. In our setting where $S = S_{1,1}$, for $Y \in \text{Out}(X, \alpha)$, this would be either a single stretch path or a union of two stretch paths $[X, Z]$ and $[Z, Y]$ where both $\Lambda(X, Z)$ and $\Lambda(Z, Y)$ contain α . Hence, each one is either a stretch path along α^+ or α^- . Assume $Z = \text{stretch}(X, \alpha^-, t_1)$ and $Y = \text{stretch}(Z, \alpha^+, t_2)$. Set $Z_t^- = \text{stretch}(Z, \alpha^-, t)$ and $Z_t^+ = \text{stretch}(Z, \alpha^+, t)$. By the

calculations of the previous section, $\tau_\alpha(Z_t^-) < \tau_\alpha(Z_t^+)$. Since $Z_t^- = X_{t+t_1}^-$ and $Z_t^+ = Y$, we have

$$\ell_\alpha(X_{t_1+t_2}^-) = \ell_\alpha(Y) \quad \text{and} \quad \tau_\alpha(X_{t_1+t_2}^-) < \tau_\alpha(Y)$$

Similarly, there are s_1 and s_2 such that $s_1 + s_2 = t_1 + t_2$, $W = \text{stretch}(X, \alpha^+, s_1)$, and $Y = \text{stretch}(W, \alpha^-, s_2)$. Then $X_{t_1+t_2}^+ = X_{s_1+s_2}^+$ and by the same argument as above

$$\ell_\alpha(X_{t_1+t_2}^+) = \ell_\alpha(Y) \quad \text{and} \quad \tau_\alpha(Y) < \tau_\alpha(X_{t_1+t_2}^+)$$

That is, Y is inside of the sector bounded by the stretch rays X_t^+ and X_t^- . The analogous statement for $\text{In}(X, \alpha)$ is proved similarly. \square

Figure 0 (on the title page) illustrates Proposition 5.4 by showing the sets $\text{In}(X, \alpha)$ in the Poincaré disk model of $\mathcal{T}(S_{1,1})$ for X the hexagonal punctured torus and for several simple curves α , including the three systoles.

Corollary 5.5. *Given $X, Y \in \mathcal{T}(S_{1,1})$, if $\Lambda(X, Y)$ is a simple closed curve, then $\text{Env}(X, Y)$ is a compact quadrilateral.*

Proof. The statement follows from Proposition 5.4 and the fact that $\text{Env}(X, Y) = \overline{\text{Out}(X, \alpha)} \cap \overline{\text{In}(Y, \alpha)}$. \square

Proposition 5.6. *In $\mathcal{T}(S_{1,1})$, the set $\text{Env}(X, Y)$ varies continuously as a function of X and Y with respect to the topology induced by the Hausdorff distance on closed sets.*

Proof. First suppose $\Lambda(X, Y)$ is a simple closed curve α . By [Thu86b, Theorem 8.4], if $X_i \rightarrow X$ and $Y_i \rightarrow Y$, then $\Lambda(X, Y)$ contains any limit point of $\Lambda(X_i, Y_i)$; thus $\Lambda(X_i, Y_i) = \alpha$ for all sufficiently large i . That is, for sufficiently large i , $\text{Env}(X_i, Y_i)$ is a compact quadrilateral bounded by segments in the foliations \mathcal{F}_α^\pm . Let Z be the *left corner* of $\text{Env}(X, Y)$, i.e. the intersection point of the leaf of \mathcal{F}_α^+ through X and the leaf of \mathcal{F}_α^- through Y . For any neighborhood U of Z , by smoothness and transversality of \mathcal{F}_α^\pm , there is a neighborhood U_X of X and a neighborhood U_Y of Y , such that for all sufficiently large i , $X_i \in U_X$, $Y_i \in U_Y$, and the leaf of \mathcal{F}_α^+ through X_i and the leaf of \mathcal{F}_α^- through Y_i will intersect in U . That is, for all sufficiently large i , the left corner of $\text{Env}(X_i, Y_i)$ lies close to the left corner of $\text{Env}(X, Y)$. A similar argument holds for the right corners. This shows $\text{Env}(X_i, Y_i)$ converges to $\text{Env}(X, Y)$.

Now suppose $\Lambda(X, Y) = \lambda$ is a maximal-recurrent lamination and $X_i \rightarrow X$ and $Y_i \rightarrow Y$. Let $\widehat{\lambda}$ be the canonical completion of λ , and let \mathcal{G} be the bi-infinite stretch line along $\widehat{\lambda}$ passing through X and Y . Also let \mathcal{G}_i and \mathcal{G}'_i be the (bi-infinite) stretch lines along $\widehat{\lambda}$ through X_i and Y_i respectively. Note that \mathcal{G}_i and \mathcal{G}'_i either coincide or are disjoint and asymptotic to \mathcal{G} in the backward direction. If they coincide, then $\Lambda(X_i, Y_i) = \lambda$ and $\text{Env}(X_i, Y_i)$ is a segment of \mathcal{G}_i . If they are disjoint, then $\mathcal{T}(S_{1,1})$ is divided into three disjoint regions. Let M_i be the closure of the region bounded by $\mathcal{G}_i \cup \mathcal{G}'_i$; see Figure 9. In the case that $\mathcal{G}_i = \mathcal{G}'_i$, set $M_i = \mathcal{G}_i$. For any geodesic L from X_i to Y_i , since $X_i, Y_i \in M_i$, if L leaves M_i , then it must cross either \mathcal{G}_i or \mathcal{G}'_i at least twice. But two points on a stretch line cannot be connected by any other geodesic in the same direction, so L must be contained entirely in M_i . Therefore, $\text{Env}(X_i, Y_i) \subset M_i$ (see Figure 9). Since \mathcal{G}_i and \mathcal{G}'_i converge to \mathcal{G} , M_i also converges to \mathcal{G} . Therefore $\text{Env}(X_i, Y_i)$ converges to a subset of \mathcal{G} . For any $Z_i \in \text{Env}(X_i, Y_i)$,

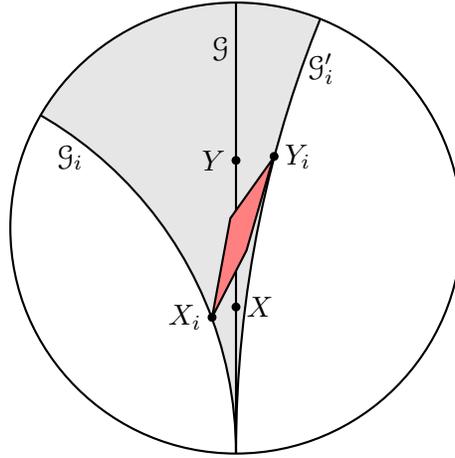


Figure 9: $\text{Env}(X_i, Y_i)$ is sandwiched between \mathcal{G}_i and \mathcal{G}'_i .

$d_{\text{Th}}(X_i, Z_i) + d_{\text{Th}}(Z_i, Y_i) = d_{\text{Th}}(X_i, Y_i)$, so by continuity of d_{Th} , Z_i must converge to a point $Z \in \mathcal{G}$ with $d_{\text{Th}}(X, Z) + d_{\text{Th}}(Z, Y) = d_{\text{Th}}(X, Y)$. In other words, Z lies on the geodesic from X to Y . This shows $\text{Env}(X_i, Y_i)$ converges to $\text{Env}(X, Y)$. \square

We can now assemble the proof of Theorem 1.1: Part (ii) is Proposition 5.6, part (iii) is Proposition 5.2(i), and part (iv) is Corollary 5.5. Part (i) (compactness of envelopes) follows, for example, from Hausdorff continuity and the compactness in the simple closed curve case established in Corollary 5.5: By [Thu86b, Theorem 10.7], any pair (X, Y) is a limit of pairs (X_i, Y_i) with $\Lambda(X_i, Y_i) = \alpha_i$ a simple curve. Thus $E(X, Y)$ is a Hausdorff limit of compact sets in $\mathcal{T}(S_{1,1})$, hence compact.

6 Thurston norm and rigidity

In this section we introduce and study Thurston's norm, which is the infinitesimal version of the metric d_{Th} , and prove Theorems 1.4–1.5.

6.1 The norm

Thurston showed in [Thu86b] that the metric d_{Th} is Finsler, i.e. it is induced by a norm $\|\cdot\|_{\text{Th}}$ on the tangent bundle. This norm is naturally expressed as the infinitesimal analogue of the length ratio defining d_{Th} :

$$\|v\|_{\text{Th}} = \sup_{\alpha} \frac{d_X \ell_{\alpha}(v)}{\ell_{\alpha}(X)} = \sup_{\alpha} d_X(\log \ell_{\alpha})(v), \quad v \in T_X \mathcal{T}(S) \quad (17)$$

In what follows we will need a result about the regularity of this norm function on the tangent bundle of $\mathcal{T}(S)$.

Theorem 6.1. *Let S be a surface of finite hyperbolic type. Then the Thurston norm function $T\mathcal{J}(S) \rightarrow \mathbf{R}$ is locally Lipschitz.*

The Thurston norm is defined as a supremum of a collection of 1-forms; as such, the following lemma allows us to deduce its regularity from that of the forms:

Lemma 6.2. *Let M be a smooth manifold and \mathcal{E} a collection of 1-forms on M . Suppose that \mathcal{E} , considered as a collection of sections of T^*M , is locally uniformly bounded and locally uniformly Lipschitz. Then the function $E : TM \rightarrow \mathbf{R}$ defined by*

$$E(v) := \sup_{e \in \mathcal{E}} e(v)$$

is locally Lipschitz (assuming it is finite at one point).

Note that “locally Lipschitz” is well-defined for real-valued functions on any smooth manifold, or sections of any vector bundle, by working in a local chart and using the standard norm on \mathbf{R}^n as a metric.

Proof. Any linear function $\mathbf{R}^n \rightarrow \mathbf{R}$ is Lipschitz, however the Lipschitz constant is proportional to its norm as an element of $(\mathbf{R}^n)^*$. Thus, for example, a family of linear functions is uniformly Lipschitz only when the corresponding subset of $(\mathbf{R}^n)^*$ is bounded.

For the same reason, if we take a family of 1-forms on M (sections of T^*M) and consider them as fiberwise-linear functions $TM \rightarrow \mathbf{R}$, then in order for these functions to be locally uniformly Lipschitz, we must require the sections to be *both* locally uniformly Lipschitz and locally bounded.

Thus the hypotheses on \mathcal{E} are arranged exactly so that the family of functions $TM \rightarrow \mathbf{R}$ of which E is the supremum is locally uniformly Lipschitz.

The supremum of a family of locally uniformly Lipschitz functions is locally Lipschitz or identically infinity. Since the function E is such a supremum, we find that it is locally Lipschitz once it is finite at one point. \square

Proof of Theorem 6.1. By (17), the Thurston norm is a supremum of the type considered in Lemma 6.2. Therefore, it suffices to show that the set

$$d\log \mathcal{C} := \{d\log \ell_\alpha : \alpha \text{ a simple curve} \}$$

of 1-forms on $\mathcal{J}(S)$ is locally uniformly bounded and locally uniformly Lipschitz.

To see this, first recall that length functions extend continuously from curves to the space $\mathcal{ML}(S)$ of measured laminations, and from Teichmüller space to the complex manifold $\mathcal{QF}(S)$ of quasi-Fuchsian representations in which $\mathcal{J}(S)$ is a totally real submanifold. The resulting length function $\ell_\lambda : \mathcal{QF}(S) \rightarrow \mathbf{C}$ for a fixed lamination $\lambda \in \mathcal{ML}(S)$ is holomorphic, and the dependence on λ is continuous in the locally uniform topology of functions on $\mathcal{QF}(S)$. Such a holomorphic extension was mentioned in [Thu86b] and formally developed by Bonahon in [Bon96, Section 11] (existence, holomorphicity) and [Bon98, pp. 20–21] (dependence on λ). For holomorphic functions, locally uniform convergence implies locally uniform convergence

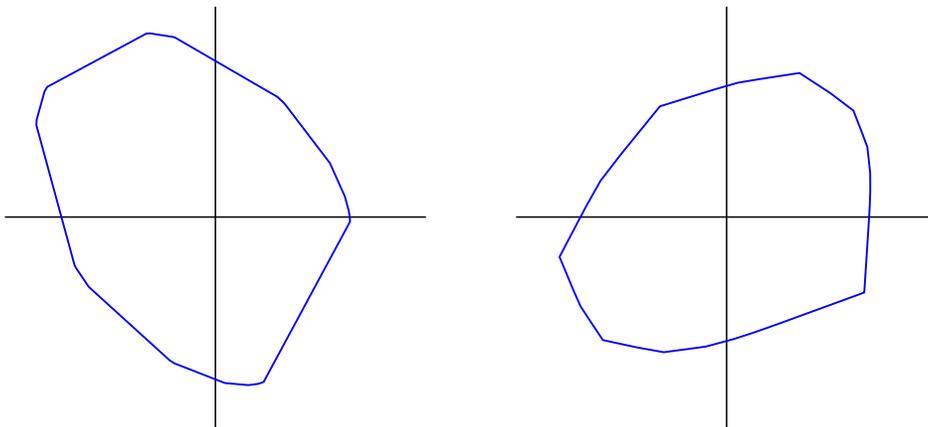


Figure 10: At left, the unit sphere $T_X^1\mathcal{T}(S_{1,1})$ of the Thurston norm on the tangent space at the point $X = 0.35 + 1.8i$ in the upper half-plane model of $\mathcal{T}(S_{1,1})$. At right, the unit sphere of the dual norm on the cotangent space.

of derivatives of any fixed order, so we find that the derivatives of ℓ_λ also depend continuously on λ .

Restricting to $\mathcal{T}(S) \subset \mathcal{QF}(S)$, and noting that the length of a nonzero measured lamination does not vanish on $\mathcal{T}(S)$, we see that the 1-form $d\log(\ell_\lambda)$ on $\mathcal{T}(S)$ is real-analytic, and that the map $\lambda \mapsto d\log(\ell_\lambda)$ is continuous from $\mathcal{ML}(S) \setminus \{0\}$ to the C^1 topology of 1-forms on any compact subset of $\mathcal{T}(S)$.

Since the 1-form $d\log \ell_\lambda$ is invariant under scaling λ , it is naturally a function (still C^1 continuous) of $[\lambda] \in \mathcal{PML}(S) = (\mathcal{ML}(S) \setminus \{0\})/\mathbf{R}^+$. Because $\mathcal{PML}(S)$ is compact, this implies that the collection of 1-forms

$$d\log \mathcal{PML} := \{d\log \ell_\lambda : [\lambda] \in \mathcal{PML}(S)\}$$

is locally uniformly bounded in C^1 . In particular it is locally uniformly Lipschitz, and since this collection contains $d\log \mathcal{C}$, we are done. \square

6.2 Shape of the unit sphere

Fix a point $X \in \mathcal{T}(S_{1,1})$ for the rest of this section. Let $T_X^1\mathcal{T}(S_{1,1})$ denote the unit sphere (circle) of Thurston's norm, i.e.

$$T_X^1\mathcal{T}(S_{1,1}) = \{v \in T_X\mathcal{T}(S_{1,1}) : \|v\|_{\text{Th}} = 1\}.$$

An example of the Thurston unit sphere and its dual are shown in Figure 10. We will show that in general, the shape of the unit sphere determines X up to the action of the mapping class group.

Let \mathcal{RL} be the set of geodesic laminations that are completions of maximal-recurrent laminations on $S_{1,1}$. Define

$$v_X : \mathcal{RL} \rightarrow T_X^1\mathcal{T}$$

to be the map that sends $\lambda \in \mathcal{RL}$ to the $\|\cdot\|_{\text{Th}}$ -unit vector tangent to the stretch path from X along λ . Recall that a simple curve α is contained in completions of maximal-recurrent laminations $\alpha^+, \alpha^- \in \mathcal{RL}$ that are obtained by adding a spiraling leaf.

Thurston showed in [Thu86b] that $T_X^1 \mathcal{J}(S_{1,1})$ is the boundary of a convex open neighborhood of the origin in $T_X \mathcal{J}(S_{1,1})$. This convex curve contains a flat segment for every curve α . We denote the set of vectors in this flat segment by $F(X, \alpha)$. That is, the set of vectors where α is maximally stretched along the associated geodesic is an interval in the unit circle with end points $v_X(\alpha^+)$ and $v_X(\alpha^-)$. We denote by $|F(X, \alpha)|$ the length of this segment with respect to $\|\cdot\|_{\text{Th}}$, i.e.

$$|F(X, \alpha)| = \|v_X(\alpha^+) - v_X(\alpha^-)\|_{\text{Th}}$$

First we need the following lemma.

Lemma 6.3. *Let $\text{EQ}_\alpha(X, t)$ be the earthquake path along α with $\text{EQ}_\alpha(X, 0) = X$. Let $\text{EQ}_\alpha = \frac{d}{dt} \text{EQ}_\alpha(t) \Big|_{t=0}$. Then we have*

$$\left\| \text{EQ}_\alpha \right\|_{\text{Th}} \stackrel{*}{\asymp}_X \ell_\alpha(X).$$

Proof. For this proof, we abbreviate $\ell_\beta(t) = \ell_\beta(E_t)$ and write $\dot{\ell}_\beta = \frac{d}{dt} \ell_\beta(t) \Big|_{t=0}$. By the definition of the Thurston norm we have

$$\left\| \text{EQ}_\alpha \right\|_{\text{Th}} = \sup_\beta \frac{|\dot{\ell}_\beta|}{\ell_\beta(X)}.$$

By Kerckhoff's formula [Ker83, Corollary 3.3],

$$\dot{\ell}_\beta = \sum_{p \in \alpha \cap \beta} \cos(\theta_p) \leq i(\alpha, \beta).$$

On the other hand, the intersection number $i(\alpha, \beta)$ is estimated by the following (see, for example, [HJ15, Proposition 3.4]):

$$i(\alpha, \beta) \stackrel{*}{\asymp}_X \ell_\alpha(X) \ell_\beta(X) \tag{18}$$

The upper bound now follows immediately.

To see the corresponding lower bound we must construct a suitable curve β . There is a constant C_0 depending on X so that we can find a set of filling curves β each having length less than C_0 so that (see [LRT12, Proposition 3.1])

$$\sum_{\beta' \in \beta} i(\beta', \alpha) \stackrel{*}{\asymp}_X \ell_X(\alpha).$$

Hence, there is a curve $\beta_0 \in \beta$ so that $i(\alpha, \beta_0) \stackrel{*}{\asymp}_X \ell_X(\alpha)$.

Let β_1 be the shortest transverse curve to β_0 and let $\beta_2 = D_{\beta_0}(\beta_1)$. Note that since the lengths of the curves β_i are bounded, there is a lower bound (depending on X) on the angle of intersection between the two of them.

Every time α intersects β_0 , it also intersects a segment of either β_1 or β_2 forming a triangle with one edge in α , one edge in β_0 and one edge in β_j , $j = 1, 2$. The angles adjacent to α cannot both be close to $\pi/2$; the sum of angle in a triangle is less than π and the angle between β_0 and β_j is bounded below. Hence, the sum of cosines of these two angles is uniformly bounded below. Thus,

$$\sum_{p \in \alpha \cap \beta_0} \cos(\theta_p) + \sum_{p \in \alpha \cap \beta_1} \cos(\theta_p) + \sum_{p \in \alpha \cap \beta_2} \cos(\theta_p) \stackrel{*}{\succ} i(\alpha, \beta_0) \stackrel{*}{\succ}_X \ell_X(\alpha).$$

Now let β be the curve β_i where $\sum_{p \in \alpha \cap \beta_i} \cos(\theta_p)$ is the largest. We have

$$\dot{\ell}_\beta = \sum_{p \in \alpha \cap \beta} \cos(\theta_p) \stackrel{*}{\succ}_X \ell_X(\alpha).$$

Hence,

$$\left\| \mathbf{EQ}_\alpha \right\|_{\text{Th}} \geq \frac{|\dot{\ell}_\beta|}{\ell_\beta(X)} \stackrel{*}{\succ}_X \ell_X(\alpha). \quad \square$$

Proposition 6.4. *For every curve α , we have*

$$|F(X, \alpha)| \stackrel{*}{\succ}_X \ell_\alpha(X)^2 e^{-\ell_\alpha(X)}.$$

Proof. Let X_t^+ and X_t^- be as in Section 5.1. These are paths with tangent vectors $v_X(\alpha^+)$ and $v_X(\alpha^-)$ respectively. Note that the length of α is the same in X_t^+ and X_t^- , hence

$$X_t^+ = \mathbf{EQ}_\alpha(X_t^-, \Delta_t), \quad (19)$$

where as before \mathbf{EQ}_α is the earthquake map along α and

$$\Delta_t = \tau_\alpha(X_t^+) - \tau_\alpha(X_t^-) = 4e^t \log \coth \frac{\ell_\alpha(X)}{2} - 4 \log \coth \frac{e^t \ell_\alpha(X)}{2}.$$

Then,

$$\left. \frac{d}{dt} \Delta_t \right|_{t=0} = 4 \log \coth \frac{\ell_\alpha(X)}{2} + 4\ell_\alpha(X) \operatorname{csch}(\ell_\alpha(X)).$$

For large values of x we have

$$\log \coth(x) \sim 2e^{-2x} \quad \text{and} \quad \operatorname{csch}(x) \sim 2e^{-x}$$

Hence for large values of $\ell_\alpha(X)$, we have

$$\left. \frac{d}{dt} \Delta_t \right|_{t=0} \sim 8e^{-\ell_\alpha(X)} + 4\ell_\alpha(X)e^{-\ell_\alpha(X)} \stackrel{*}{\succ} \ell_\alpha(X)e^{-\ell_\alpha(X)}.$$

Also, by Lemma 6.3

$$\left\| \mathbf{EQ}_\alpha \right\|_{\text{Th}} \stackrel{*}{\succ}_X \ell_\alpha(X).$$

The proposition now follows by differentiating (19) at $t = 0$ and applying the chain rule. \square

Theorem 6.5. *Let α and β be curves with $i(\alpha, \beta) = 1$. Let $\beta_n = D_\alpha^n(\beta)$. Then*

$$\lim_{n \rightarrow \infty} \frac{|\log |F(X, \beta_n)||}{n} = \ell_\alpha(X).$$

Proof. For large values of n ,

$$\ell_{\beta_n}(X) \stackrel{\pm}{\asymp} n\ell_\alpha(X).$$

The theorem now follows from Proposition 6.4. \square

Using the results above we can now show that the shape of the unit sphere in $T_X^1\mathcal{J}(S_{1,1})$ determines X up to the action of the mapping class group.

Proof of Theorem 1.4. Within the convex curve $T_X^1\mathcal{J}(S_{1,1})$ let U_0 denote an open arc which has $v_X(\alpha^-)$ as one endpoint and which is disjoint from $F(X, \alpha)$. Let v_0 denote the other endpoint of U_0 . We use “interval notation” to refer to open arcs within U_0 , where (a, b) refers to the open arc with endpoints a, b . Thus for example $U_0 = (v_X(\alpha^-), v_0)$.

Let γ_0 be a simple curve on $S_{1,1}$ so that among all of the flat segments of $T_X^1\mathcal{J}(S_{1,1})$ contained in U_0 , the segment $F(X, \gamma_0)$ has the largest length. (Here, as in the rest of the proof, “length” refers to $\|\cdot\|_{\text{Th}}$.) Now, for $i = 1, 2, 3, \dots$, let

$$U_i = (v_X(\alpha^-), v_X(\gamma_{i-1}^+)) \subset U_{i-1}$$

and let γ_i be the curve so that $F(X, \gamma_i)$ is a longest flat segment in U_i .

Assuming $\ell_\alpha(X)$ and i are large enough, we show that $i(\alpha, \gamma_i) = 1$ and $\gamma_{i+1} = D_\alpha(\gamma_i)$. To see this, first note that the twisting about α increases along the boundary as one approaches α^- . More precisely, for any two curves η_0 and η_1 , if

$$F(X, \eta_0) \subset (v_X(\alpha^-), v_X(\eta_1^+))$$

then $\text{twist}_\alpha(X, \eta_0) \geq \text{twist}_\alpha(X, \eta_1)$. Now, let $m = \text{twist}_\alpha(X, \gamma_i)$. Since we are considering large i , we can assume m is positive and large. We have, for any other curve γ ,

$$\text{twist}_\alpha(X, \gamma) \geq (m + 1) \implies F(X, \gamma) \subset U_i$$

and

$$F(X, \gamma) \subset U_i \implies \text{twist}_\alpha(\gamma, X) \geq m.$$

Since m is large, we also have

$$\frac{\ell_\gamma(X)}{i(\gamma, \alpha)} \stackrel{\pm}{\asymp} \ell_\alpha(X) \cdot \text{twist}_\alpha(\gamma, X).$$

Hence, the shortest such curve γ is one that intersects α minimally, namely once, and twists the smallest amount possible. Let γ be the shortest curve with $F(X, \gamma) \subset U_i$ and let γ' be any other curve with $F(X, \gamma') \subset U_i$. If $i(\gamma', \alpha) \geq 2$ then, up to an additive error, γ' is at least twice as long as γ ; if it twists more, then

$$\ell_{\gamma'}(X) \stackrel{\pm}{\asymp} \ell_\gamma(X) + \ell_\alpha(X).$$

By Proposition 6.4, $|F(X, \gamma')|$ is smaller than $|F(X, \gamma)|$ by at least a factor comparable to $\ell_X(\alpha)$. Hence, when $\ell_X(\alpha)$ is large enough the segment $F(X, \gamma)$ is the longest segment in U_i , which means that $\gamma_{i+1} = \gamma$.

This means that for large i , the curve γ_i intersects α only once. In fact, the above argument shows that if $i(\alpha, \gamma_i) = 1$ then $\gamma_{i+1} = D_\alpha(\gamma_i)$. That is, for large i , $\gamma_i = \beta_i$ where β_i is as in Theorem 6.5. Applying Theorem 6.5, we see that the length of α itself is determined by the asymptotic behavior of the lengths $|F(\alpha, \gamma_i)|$.

An argument very similar to the one above shows that the longest flat segment in a small neighborhood of $v_X(\alpha^+)$ is a large *negative* Dehn twist of a curve intersecting α once, and that through the asymptotics of their lengths, the geometry of the norm sphere near $v_X(\alpha^+)$ also determines the length of α . As before this applies to any simple curve α that is sufficiently long on X . Collectively, we refer to the arguments above as the *longest segment construction*.

Now for $X, Y \in \mathcal{T}(S_{1,1})$, assume that there is a norm preserving linear map

$$L: T_X \mathcal{T}(S_{1,1}) \rightarrow T_Y \mathcal{T}(S_{1,1}).$$

Then, the flat segments of $T_X^1 \mathcal{T}(S_{1,1})$ are mapped bijectively to those of $T_Y^1 \mathcal{T}(S_{1,1})$. For any simple curve γ we denote by γ^* the simple curve such that $L(F(X, \gamma)) = F(Y, \gamma^*)$.

Choose a simple curve α so that $\ell_X(\alpha)$ and $\ell_Y(\alpha^*)$ are large enough so that the longest segment construction applies to both of them. Then we obtain a sequence of curves $\gamma_i = D_\alpha^i \beta$ which satisfy $i(\gamma_i, \alpha) = 1$, and whose segments $F(X, \gamma_i)$ approach one endpoint of $F(X, \alpha)$ with each being longest in some neighborhood of that endpoint. As L is an isometry, the image segments $F(X, \gamma_i^*)$ approach some endpoint of $F(X, \alpha^*)$ and are locally longest in the same sense. Thus the curves γ_i^* are also obtained by applying powers (positive or negative) of a Dehn twist about α^* to a fixed curve and they satisfy $i(\gamma_i^*, \alpha^*) = 1$. Since $|F(X, \gamma_i)| = |F(Y, \gamma_i^*)|$ we conclude $\ell_X(\alpha) = \ell_X(\alpha^*)$.

Now choose an integer N so that $\ell_X(\gamma_N)$ and $\ell_Y(\gamma_N^*)$ are large enough to apply the longest segment construction (to γ_N and γ_N^* , respectively). Proceeding as in the previous paragraph, we find $\ell_X(\gamma_N) = \ell_Y(\gamma_N^*)$.

At this point we have two pairs of simple curves intersecting once, (α, γ_N) and (α^*, γ_N^*) , and the lengths of the first pair on X are equal to those of the second pair on Y . This implies that X and Y are in the same orbit of the extended mapping class group: Take a mapping class ϕ with $\phi(\alpha) = \alpha^*$ and $\phi(\gamma_N) = \gamma_N^*$. Then α has the same length on X and $\phi^{-1}(Y)$, so these points differ only in the Fenchel-Nielsen twist parameter (relative to pants decomposition α). Since the length of γ_N is the same as well, either the twist parameters are equal and $X = \phi^{-1}(Y)$ or the twist parameters differ by a sign and $X = r(\phi^{-1}(Y))$ where r is the orientation-reversing mapping class which preserves both α and γ_N while reversing orientation of γ_N . \square

6.3 Local and global isometries

Before proceeding with the proof of Theorem 1.5 we recall some standard properties of the extended mapping class group action on $\mathcal{T}(S_{1,1})$. (For further discussion, see for example

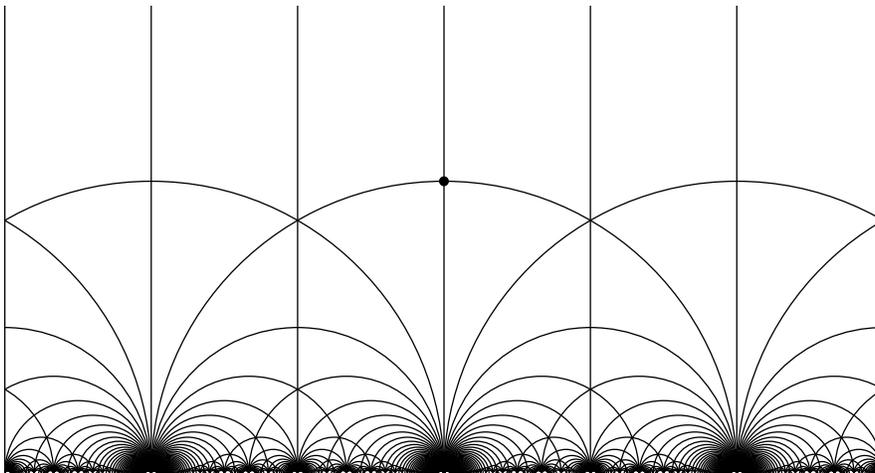


Figure 11: The standard $(2,3,\infty)$ triangle tiling of the upper half plane. The marked point is the imaginary unit i .

[Kee74, Section 2] [FM12, Section 2.2.4].)

The mapping class group $\text{Mod}(S_{1,1}) = \text{Homeo}^+(S_{1,1})/\text{Homeo}_0(S_{1,1})$ of the punctured torus is isomorphic to $\text{SL}(2, \mathbf{Z})$, and identifying $\mathcal{T}(S_{1,1})$ with the upper half-plane \mathbf{H} in the usual way, the action of $\text{Mod}(S_{1,1})$ becomes the action of $\text{SL}(2, \mathbf{Z})$ by linear fractional transformations. Similarly, the extended mapping class group $\text{Mod}^\pm(S_{1,1}) = \text{Homeo}(S_{1,1})/\text{Homeo}_0(S_{1,1})$ can be identified with $\text{GL}(2, \mathbf{Z})$, where an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant -1 acts on \mathbf{H} by the conjugate-linear map $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$. Neither of these groups acts effectively on \mathbf{H} , since in each case the elements $\pm I$ act trivially; thus when considering the action on $\mathcal{T}(S_{1,1})$ it is convenient to work with the quotients $\text{PSL}(2, \mathbf{Z})$ and $\text{PGL}(2, \mathbf{Z})$ which act effectively.

The properly discontinuous action of $\text{PGL}(2, \mathbf{Z})$ on \mathbf{H} preserves the standard $(2,3,\infty)$ triangle tiling of \mathbf{H} (see Figure 11), with the cells of each dimension in this tiling corresponding to different types of isotropy; specifically, we have:

- A point in the interior of a triangle has trivial stabilizer in $\text{PGL}(2, \mathbf{Z})$.
- A point in the interior of an edge has stabilizer in $\text{PGL}(2, \mathbf{Z})$ isomorphic to $\mathbf{Z}/2$ and generated by a *reflection*, i.e. an element conjugate to a either $z \mapsto -\bar{z}$ or $z \mapsto -\bar{z} + 1$.

While vertices of the tiling have larger stabilizers, the only property of such points we will use is that they form a discrete set.

Proof of Theorem 1.5. Let U be an open connected set in $\mathcal{T}(S_{1,1})$ and let $f : (U, d_{\text{Th}}) \rightarrow (\mathcal{T}(S_{1,1}), d_{\text{Th}})$ be an isometric embedding.

By Theorem 6.1, the Thurston norm is locally Lipschitz (locally $C_{loc}^{0,1}$). By [MT16, Theorem A] an isometry of such Finsler spaces is $C_{loc}^{1,1}$ and its differential is norm-preserving. Therefore, for each $X \in U$ the differential

$$d_X f : T_X \mathcal{T}(S_{1,1}) \rightarrow T_{f(X)} \mathcal{T}(S_{1,1})$$

is an isometry for the Thurston norm and by Theorem 1.4 there exists $\Phi(X) \in \mathrm{PGL}(2, \mathbf{Z})$ such that

$$f(X) = \Phi(X) \cdot X \quad (20)$$

This property may not determine $\Phi(X) \in \mathrm{PGL}(2, \mathbf{Z})$ uniquely, however, choosing one such element for each point of U we obtain a map $\Phi : U \rightarrow \mathrm{PGL}(2, \mathbf{Z})$.

Let $X_0 \in U$ be a point with trivial stabilizer in $\mathrm{PGL}(2, \mathbf{Z})$. Using proper discontinuity of the $\mathrm{PGL}(2, \mathbf{Z})$ action, we can select neighborhoods V of X_0 and W of $f(X_0)$ so that

$$\{\phi \in \mathrm{PGL}(2, \mathbf{Z}) : \phi \cdot V \cap W \neq \emptyset\} = \{\Phi(X_0)\}$$

However, by continuity of f and (20) we find that the $\Phi(X)$ is an element of this set for all X near X_0 . That is, the map Φ is locally constant at X_0 . More generally, this shows Φ is constant on any connected set consisting of points with trivial stabilizer.

Now we consider the behavior of Φ and f in a small neighborhood V of a point X_1 with $\mathbf{Z}/2$ stabilizer—that is, a point in the interior of an edge e of the $(2, 3, \infty)$ triangle tiling. Taking V to be a sufficiently small disk, we can assume $V \setminus e$ has two components, which we label by V_{\pm} , and that each component consists of points with trivial stabilizer (equivalently, V does not contain any vertices of the tiling). By the discussion above Φ is constant on V_+ and on V_- , and we denote the respective values by ϕ_+ and ϕ_- . By continuity of f , the element $\phi_+^{-1}\phi_- \in \mathrm{PGL}(2, \mathbf{Z})$ fixes $e \cap V$ pointwise and is therefore either the identity or a reflection. In the latter case f would map both sides of e (locally, near X_1) to the same side of the edge $f(e)$, and hence it would not be an immersion at X_1 . This is a contradiction, for we have seen that the differential of f is an isomorphism at each point. We conclude $\phi_+ = \phi_-$, and f agrees with this extended mapping class on $V \setminus e$. By continuity of f the same equality extends over the edge e .

Let $U' \subset U$ denote the subset of points with trivial or $\mathbf{Z}/2$ stabilizer. We have now shown that for each $X \in U'$ there exists a neighborhood of X on which f is equal to an element of $\mathrm{PGL}(2, \mathbf{Z})$. An element of $\mathrm{PGL}(2, \mathbf{Z})$ is uniquely determined by its action on any open set, so this local representation of f by a mapping class is uniquely determined and locally constant. Thus on any connected component of U' we have that f is equal to a mapping class. However U' is connected, since U is connected and open and the set of points in $\mathcal{T}(S_{1,1})$ with larger stabilizer (i.e. the vertex set of the tiling) is discrete.

We have therefore shown $f = \phi$ on U' , for some $\phi \in \mathrm{PGL}(2, \mathbf{Z})$. Finally, both f and ϕ are continuous, and U' is dense in U , equality extends to U , as required. \square

References

- [Bon96] F. Bonahon. Shearing hyperbolic surfaces, bending pleated surfaces and Thurston's symplectic form. *Ann. Fac. Sci. Toulouse Math. (6)*, 5(2):233–297, 1996.
- [Bon98] F. Bonahon. Variations of the boundary geometry of 3-dimensional hyperbolic convex cores. *J. Differential Geom.*, 50(1):1–24, 1998.

- [CR07] Y.-E. Choi and K. Rafi. Comparison between Teichmüller and Lipschitz metrics. *J. Lond. Math. Soc. (2)*, 76(3):739–756, 2007.
- [CRS08] Y.-E. Choi, K. Rafi, and C. Series. Lines of minima and Teichmüller geodesics. *Geom. Funct. Anal.*, 18(3):698–754, 2008.
- [DGK16] J. Danciger, F. Guéritaud, and F. Kassel. Geometry and topology of complete Lorentz spacetimes of constant curvature. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(1):1–56, 2016.
- [FM12] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.
- [GK] F. Guéritaud and F. Kassel. Maximally stretched laminations of geometrically finite hyperbolic manifolds. To appear in *Geom. Topol.* [arXiv:math.GT/1307.0250](https://arxiv.org/abs/math/1307.0250).
- [Gué15] F. Guéritaud. Lengthening deformations of singular hyperbolic tori. *Annales de la faculté des sciences de Toulouse*, 24(5):1239–1260, 2015.
- [Gué16] F. Guéritaud. Personal communication, 2016.
- [HJ15] Y. A. Herrera Jaramillo. Intersection numbers of geodesic arcs. *Rev. Colombiana Mat.*, 49(2):307–319, 2015.
- [Iva97] N. V. Ivanov. Automorphism of complexes of curves and of Teichmüller spaces. *Internat. Math. Res. Notices*, 14:651–666, 1997.
- [Kas09] F. Kassel. Quotients compacts d’espaces homogènes réels ou p-adiques. Ph.D. Thesis, Université Paris-Sud 11, 2009.
- [Kee74] L. Keen. On fundamental domains and the Teichmüller modular group. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 185–194. Academic Press, New York, 1974.
- [Ker83] S. P. Kerckhoff. The Nielsen realization problem. *Ann. of Math. (2)*, 117(2):235–265, 1983.
- [LRT12] A. Lenzhen, K. Rafi, and J. Tao. Bounded combinatorics and the Lipschitz metric on Teichmüller space. *Geom. Dedicata*, 159:353–371, 2012.
- [LRT15] A. Lenzhen, K. Rafi, and J. Tao. The shadow of a Thurston geodesic to the curve graph. *J. Topol.*, 8(4):1085–1118, 2015.
- [Min96] Y. N. Minsky. Extremal length estimates and product regions in Teichmüller space. *Duke Math. J.*, 83(2):249–286, 1996.
- [Min99] Y. N. Minsky. The classification of punctured-torus groups. *Ann. of Math. (2)*, 149(2):559–626, 1999.
- [MM00] H.A. Masur and Y. N. Minsky. Geometry of the complex of curves. II. Hierarchical structure. *Geom. Funct. Anal.*, 10(4):902–974, 2000.
- [MT16] V. Matveev and M. Troyanov. The Myers-Steenrod theorem for Finsler manifolds of low regularity. Preprint, 2016. [arXiv:1605.03850](https://arxiv.org/abs/1605.03850).

- [PT07] A. Papadopoulos and G. Th  ret. On Teichm  ller’s metric and Thurston’s asymmetric metric on Teichm  ller space. In *Handbook of Teichm  ller theory. Vol. I*, volume 11 of *IRMA Lect. Math. Theor. Phys.*, pages 111–204. Eur. Math. Soc., Z  rich, 2007.
- [Raf05] K. Rafi. A characterization of short curves of a Teichm  ller geodesic. *Geometry and Topology*, 9:179–202, 2005.
- [Raf14] K. Rafi. Hyperbolicity in Teichm  ller space. *Geom. Topol.*, 18(5):3025–3053, 2014.
- [Roy71] H. L. Royden. Automorphisms and isometries of Teichm  ller space. In *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 369–383. Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.
- [Th  07] G. Th  ret. On the negative convergence of Thurston’s stretch lines towards the boundary of Teichm  ller space. *Ann. Acad. Sci. Fenn. Math.*, 32(2):381–408, 2007.
- [Thu86a] W. P. Thurston. Geometry and topology of 3–manifolds. Princeton University Lecture Notes, 1986. Available at www.msri.org/publications/books/gt3m.
- [Thu86b] W. P. Thurston. Minimal stretch maps between hyperbolic surfaces. Preprint, 1986. [arXiv:math.GT/9801039](https://arxiv.org/abs/math.GT/9801039).
- [Wal14] C. Walsh. The horoboundary and isometry group of Thurston’s Lipschitz metric. In *Handbook of Teichm  ller theory. Vol. IV*, volume 19 of *IRMA Lect. Math. Theor. Phys.*, pages 327–353. Eur. Math. Soc., Z  rich, 2014.

Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL, USA
david@dumas.io

Department of Mathematics
University of Rennes
Rennes, FR
anna.lenzhen@univ-rennes1.fr

Department of Mathematics
University of Toronto
Toronto, CA
rafi@math.toronto.edu

Department of Mathematics
University of Oklahoma
Norman, OK, USA
jing@math.ou.edu