THE CONFORMAL GEOMETRY OF BILLIARDS

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1. Introduction

In this note, we examine the dynamics of billiards on polygonal tables. This is intended to be neither new research nor a survey, but rather a snapshot of recent work in one corner of the billiard-dynamics arena. We will concentrate on billiard tables where all interior angles are rational multiples of π. This class of billiard tables is closely related to the study of *translation surfaces*, Riemann surfaces X equipped with a holomorphic 1-form ω, thus endowing X with a flat Euclidean metric structure away from finitely many cone-type singularities. Many recent results about billiard tables of this type come from general statements about moduli spaces of translation surfaces. The theme of this note is the search for *dynamically optimal* billiard tables: tables on which any billiard trajectory (which avoids the corners) is either periodic or it covers the table uniformly. Figure 1.1 shows an example of a dynamically optimal table. Careful definitions and examples are given in later sections; the following is an overview of the presentation.

A polygonal billiard table T, with all angles equal to rational multiples of π, gives rise to

- a translation surface (X_T, ω_T) with genus $g(X_T) \geq 1$, via a process called *unfolding*; and
- a discrete subgroup $\Gamma_T \subset \mathrm{SL}_2\mathbb{R}$, the stabilizer of (X_T, ω_T) under a *stretching* operation.

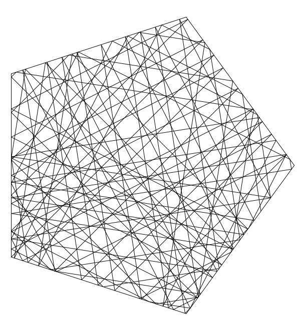

Figure 1.1. A billiard trajectory on the regular pentagon.

The genus $g(X_T)$ is easily computable from the table T; the formula is stated below in equation (3.1). Every table with genus $g(X_T) = 1$ has optimal dynamics as we discuss in §4.2. In genus 2, by studying the orbits of an $\mathrm{SL}_2\mathbb{R}$-action on the moduli space of translation surfaces, McMullen showed:

Theorem 1.1. [Mc2] *For $g(X_T) = 2$, a table T has optimal dynamics if and only if Γ_T is a lattice in $\mathrm{SL}_2\mathbb{R}$.*

Billiard tables with Γ_T a lattice are in fact quite rare. McMullen established a complete list of tables T with optimal dynamics in genus 2 [Mc3]; see Theorem 4.3.

It was first observed by Veech that for every genus $g(X_T)$, if Γ_T is a lattice in $\mathrm{SL}_2\mathbb{R}$, then the table T has optimal dynamics [Ve]. In fact, his statement was much stronger: the geodesic flow on a translation surface (X, ω) (which is not necessarily the translation surface for a billiard table) is dynamically optimal whenever Γ is a lattice, where $\Gamma = \Gamma(X, \omega)$ is the so-called *Veech group* of the surface. Theorem 1.1 is itself a consequence of a more general statement about the geodesic flow on translation surfaces of genus 2; see §4. It is reasonable to guess that the equivalence of Theorem 1.1 holds for translation surfaces in every genus. However, Smillie and Weiss have shown recently:

Theorem 1.2. [SW] *There exist translation surfaces (X, ω) which have optimal flow dynamics but for which the Veech group Γ is not a lattice.*

Theorem 1.2 leaves open the existence of billiard tables with optimal dynamics but non-lattice Veech group; billiard surfaces (X_T, ω_T) and their $\mathrm{SL}_2\mathbb{R}$-orbits form only a small (measure 0) subset of the moduli space of translation surfaces in any genus > 1.

The Smillie-Weiss examples rely on a covering construction of Hubert and Schmidt [HS]: there exist surfaces (X, ω) with lattice Veech group and holomorphic branched coverings of finite degree

$$f : Y \to X$$

so that the Veech group of the translation surface $(Y, f^*\omega)$ is not a lattice nor even finitely generated. In certain cases, the dynamical properties of the geodesic flow on the surface (X, ω) are preserved when passing to the branched cover.

The Hubert-Schmidt construction has led to further collections of interesting examples. We conclude this note with a discussion of the following recent result of Cheung, Hubert, and Masur:

Theorem 1.3. [CHM] *The billiard dynamics on the isosceles triangle with angles $(2\pi/5, 3\pi/10, 3\pi/10)$ satisfy a topological dichotomy but are non-optimal: for each direction, either all billiard trajectories are closed or all are dense, but there exist directions in which billiard trajectories are dense but not uniformly distributed.*

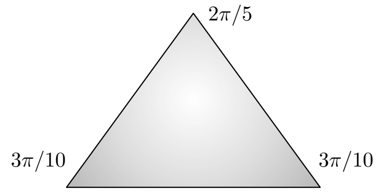

Figure 1.2. A triangular table with non-optimal dynamics but satisfying the topological dichotomy.

In standard dynamical language, the billiard flow in each direction is either completely periodic or minimal, while certain minimal directions are not uniquely ergodic. The triangle of Theorem 1.3 is the only known billiard table with this type of dynamics.

Acknowledgements. I am greatly indebted to the experts in translation surfaces for helping me prepare this note. Special thanks go to Yitwah Cheung, Howard Masur, and Curt McMullen for the conversations and numerous email exchanges about their work. Also, Curt McMullen generated the images of Figures 1.1 and 2.2. I would like to thank Jayadev Athreya, Matthew Bainbridge, Alex Eskin, and John Smillie for helpful and lengthy discussions about other recent results in this area. This note does not do justice to their beautiful mathematics. I thank the AMS for giving me this opportunity to learn about billiards; my research is also supported by the National Science Foundation and the Sloan Foundation.

2. Billiard tables

For this article, a *billiard table* means a polygon in $\mathbb{R}^2$ with all angles a rational multiple of π. See Figure 2.1. A *billiard trajectory* in direction θ is a straight-line path which begins at some point in the interior of the table, at angle θ as measured from the positive real axis, and bounces off the edges with angle of reflection equal to the angle of incidence. If a billiard trajectory hits a vertex of the polygon, it stops. As the angles are rational multiples of π, the billiard path will travel again in direction θ after finitely many reflections off the sides of the table.

2.1. **The square table.** The simplest example is the square table of side length 1, with sides parallel to the coordinate axes. In this table, it is easy to see that any billiard trajectory of slope p/q for integers p and q will either hit a vertex or eventually return to its original configuration (position and angle). If the trajectory encounters a vertex, then it must encounter a vertex also in backward time (traveling in the opposite direction). On the other hand, for all irrational slopes and any initial point, if the trajectory encounters a vertex then it will never hit a vertex in backwards time; all infinite trajectories bounce around the table spending equal time in parts with equal area.

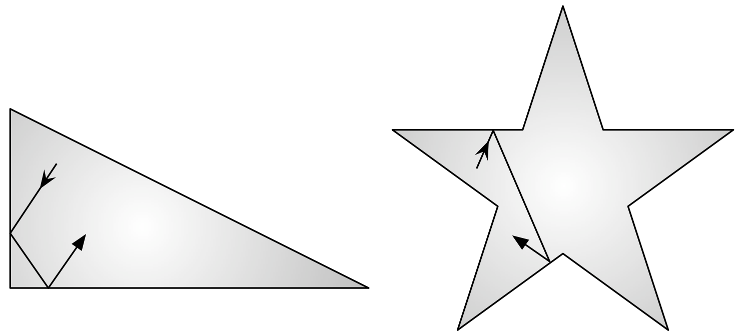

FIGURE 2.1. Polygonal billiard tables and trajectories.

2.2. **Optimal billiard dynamics.** We say a billiard table has *optimal dynamics* if for each direction θ, one of the following holds:

(1) every trajectory is either periodic or encounters a vertex in both forward and backward time; or
(2) every trajectory is infinite in either forward or backward time (or both), and every infinite trajectory is uniformly distributed.

We must take care in our meaning of uniform distribution. Because the table has rational angles, every trajectory points in only finitely many different directions under reflections off the sides of the tables. For each direction θ, we may take multiple copies of the polygonal table, one for each direction arising by reflection of trajectories in direction θ. We say a trajectory is uniformly distributed if it equidistributes with respect to Lebesgue measure on this union of tables. In other words, for any trajectory of infinite length in direction θ, let $\gamma(t)$, $t \geq 0$, be a parametrization of this trajectory with unit speed (so with each reflection in a side, $\gamma(t)$ jumps to another copy of the table). For each time $s > 0$, we can define probability measure on the union of tables by

$$\frac{1}{s}\,\gamma_* m_s$$

where m_s is arc-length measure on the interval $[0, s]$ in $\mathbb{R}$. Uniform distribution means that this family of measures converges weakly as $s \to \infty$ to normalized area measure on the finite union of polygonal tables. The property of optimal dynamics is also called *Veech dichotomy* in the literature.

2.3. **Examples and non-examples.** As explained in §2.1, the unit square table has optimal dynamics. In fact, any polygon which is tiled by a square (so that all vertices of the polygon coincide with vertices of the square tiles) will also have optimal dynamics [GJ]. A distinctly different class of examples was studied by Veech, who showed:

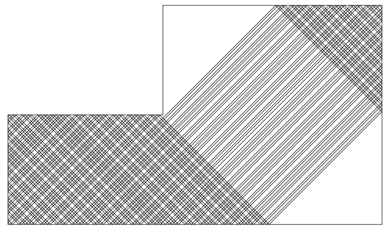

FIGURE 2.2. A billiard trajectory on an L-shaped table, neither closed nor dense.

Theorem 2.1. [Ve] *For every $n \geq 3$, the regular n-gon is a dynamically optimal billiard table.*

On the other hand, it is easy to construct tables with non-optimal dynamics. For example, begin with a square table and attach a rectangle to one side with side lengths a and b where $a/b \notin \mathbb{Q}$. See Figure 2.2. Taking the direction $\theta = \pi/4$, we see that the trajectories in direction θ which enter the smaller rectangle are neither closed nor dense.

2.4. **Topologically optimal tables.** There is a notion called topological dichotomy for billiard tables which is weaker than optimal dynamics. A billiard table satisfies the topological dichotomy if for each direction θ,

(1) every trajectory is either periodic or encounters a vertex in both forward and backward time; or
(2) every trajectory is infinite in either forward or backward time (or both), and every infinite trajectory is dense.

As with uniform distribution, we require that the trajectory be dense on the finite union of tables corresponding to different directions under reflection.

It is a non-trivial task to find billiard table examples which have dense but non-uniformly distributed trajectories. The following examples were studied by Masur and Smillie, following a construction of Veech; see [MT], [MS]. Consider a rectangular table with barrier as in Figure 2.3: begin with a rectangular table of side lengths 1 and 2 and build a perpendicular wall in the middle of the long side of length $\ell < 1$. When ℓ is rational, the table is dynamically optimal. When ℓ is irrational, the table is neither dynamically optimal nor topologically optimal, and it has billiard trajectories which are dense and non-uniformly distributed. When ℓ is Diophantine (so it is not too closely approximated by rationals), the set of directions $\theta \in [0, 2\pi)$ with dense but non-uniformly distributed trajectories is as large as possible, having Hausdorff dimension 1/2 [Ch].

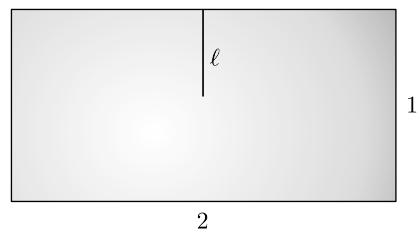

FIGURE 2.3. A rectangular billiard table with barrier of length ℓ.

3. THE TRANSLATION SURFACE OF A BILLIARD TABLE

In this section we describe the process of *unfolding*, passing from a polygonal billiard table to a Riemann surface equipped with a holomorphic 1-form. In this way, a billiard trajectory which bounces off the walls of the table *unfolds* into a straight line on the surface.

3.1. **Unfolding a billiard table.** Fix a polygon T in $\mathbb{R}^2$ and assume that all of its angles are rational multiples of π. Let $G \subset \mathrm{O}_2(\mathbb{R})$ be the group generated by reflections in the sides of T. Because of the rational angles, the group G is finite; let $N = |G|$. If the interior angles of T are expressed as $m_i\pi/n_i$, where the integers m_i and n_i have no common factors, then the number N is equal to twice the least common multiple of the n_i.

Take N copies of T, one for each reflected image gT with $g \in G$. Glue edges of distinct copies according to the reflection rules: if $h \in G$ is represented by reflection across an edge e of gT, then e is glued to its image in hgT. The genus of the resulting surface X_T is given by the formula

$$g(X_T) = 1 + \frac{N}{4}\left(k - 2 - \sum_{i=1}^{k} \frac{1}{n_i}\right) \tag{3.1}$$

where k is the number of vertices of T; see [MT].

It is easy to see that the unit square unfolds into a torus, as does an equilateral triangle. For the regular pentagon depicted in Figure 1.1, the reflection group has 10 elements, and the table unfolds into a surface of genus 6. On the other hand, the $(2\pi/5, 3\pi/10, 3\pi/10)$ triangle shown in Figure 1.2 tiles the regular pentagon, but it unfolds into a surface of genus 4.

The Euclidean coordinates on the polygon T induce a flat conformal structure on the resulting surface, together with a finite collection of cone-point singularities (where the total angle at a point exceeds 2π). This structure can be recorded by the holomorphic 1-form dz on T, glued up to define a 1-form ω_T on the unfolded surface

X_T. The cone points are simply the zeroes of ω_T. The pair (X_T, ω_T) defines the *translation surface* associated to the table T.

In fact, every compact Riemann surface X equipped with a holomorphic 1-form can be obtained by gluing polygons in this way, though the polygons will not generally be reflections of a single polygonal shape T; see the discussion in [Ma2].

3.2. **Geodesic flow on a translation surface.** The notions of trajectories and optimal dynamics can be defined on general translation surfaces (X, ω). Indeed, the 1-form ω gives a natural way to choose local Euclidean coordinates on X away from the zeroes of ω. Namely, for any point $z_0 \in X$ with $\omega_{z_0} \neq 0$, we can integrate ω to define a coordinate chart near z_0 by

$$\varphi(z) = \int_{z_0}^{z} \omega$$

which is locally invertible and locally independent of path. In fact, the transition functions for these coordinate charts are given by translations, which explains the term "translation" surface. The Euclidean charts induce a flat metric on the surface, away from the zeroes of ω, and the geodesics in this metric are simply the straight lines in these coordinates. The charts glue up at the zeroes of ω to form the cone-like singularities.

When a surface comes from unfolding a billiard table, the straight lines are precisely the unfolded billiard trajectories. Thus, we can discuss the geodesics on a general translation surface to make conclusions about billiard trajectories. We say a translation surface (X, ω) has *optimal flow dynamics* if its geodesics in each direction satisfy the dichotomy of §2.2. We say the surface satisfies the *topological dichotomy* if its geodesics in each direction satisfy the dichotomy of §2.4.

3.3. **Most geodesics are uniformly distributed.** The idea of unfolding seems to have first appeared in [KZ], where the authors studied topological transitivity of the geodesic flow on the associated translation surface, concluding that most directions on a table are minimal (all orbits are dense). General results about differentials on closed surfaces imply that the periodic directions (where all geodesics are either closed or travel between zeroes of ω) are dense in the circle; see [MT]. On the other hand, almost every direction gives rise to uniformly distributed geodesics [KMS]. Back on the table T, we get uniform distribution of the billiard trajectories in those directions.

4. Stretching the billiard tables

In this section, we describe the stretching deformation of billiard tables and translation surfaces, and we define the Veech group associated to a table. We conclude the section with McMullen's classification of dynamically optimal billiard tables which unfold into surfaces of genus 2.

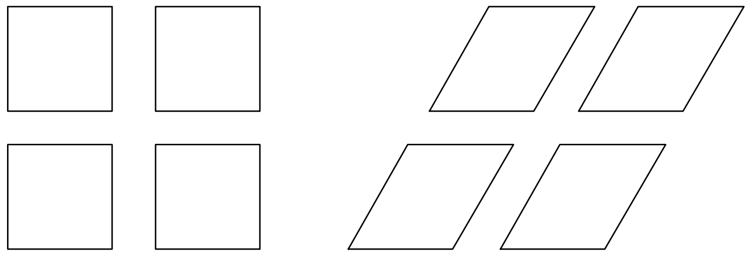

FIGURE 4.1. The square table with its reflected copies, and their images under shearing.

4.1. **An action of** $\mathrm{SL}_2\mathbb{R}$**.** The group $\mathrm{SL}_2\mathbb{R}$ consists of 2×2 matrices with real entries and determinant 1. These linear transformations act on the set of polygons in the plane: for a polygon $P \subset \mathbb{R}^2$, the matrix $A \in \mathrm{SL}_2\mathbb{R}$ sends P to the new polygon $A(P)$. This action induces a deformation of billiard tables and associated translation surfaces: let a matrix A act on each of the reflected copies of T, and glue the stretched polygons according to the same rules. The result is a family of translation surfaces (X_T^A, ω_T^A) parametrized by $A \in \mathrm{SL}_2\mathbb{R}$. See Figure 4.1.

A stretched surface (X_T^A, ω_T^A) is conformally isomorphic to (X_T, ω_T) if there exists a biholomorphic map $c : X_T^A \to X_T$ which pulls ω_T back to ω_T^A. This can be seen in terms of the polygons: the surfaces are isomorphic if the N polygons making up $A\cdot T$ can be cut into smaller polygons and reglued (without violating the reflection rules) to obtain (X, ω). If so, we say that A lies in the *Veech group* Γ_T. As an example, the Veech group of the square table is the lattice $\mathrm{SL}_2\mathbb{Z}$ consisting of all 2×2 matrices with integer entries and determinant 1.

This stretching action on billiard tables extends to an action of $\mathrm{SL}_2\mathbb{R}$ on all of $\Omega^1\mathcal{M}_g$, the moduli space of translation surfaces, and the *Veech group* $\Gamma(X, \omega)$ is the stabilizer of (X, ω). The action is again defined by the linear stretching of polygons: as mentioned before, any translation surface can be represented by a finite collection of polygons in the plane, with parallel sides glued by a translation, equipped with the 1-form dz. The Veech group $\Gamma(X, \omega)$ is easily seen to be a discrete subgroup of $\mathrm{SL}_2\mathbb{R}$, but it is a lattice only in special cases; see [Ve], [MT]. The parametrization of an orbit $\mathrm{SL}_2\mathbb{R} \to \Omega^1\mathcal{M}_g$ descends to a map $\mathbb{H} \to \mathcal{M}_g$ which is a local isometry with respect to the Poincaré metric on the upper half-plane $\mathbb{H}$ and the Teichmüller metric on $\mathcal{M}_g$; see e.g. [KMS], [Ve], [Mc1]. Translation surfaces with lattice Veech group then correspond to the so-called *Teichmüller curves*, isometrically embedded algebraic curves in $\mathcal{M}_g$.

4.2. **Genus 1.** There is a unique holomorphic 1-form (up to scaling) on a torus, coming from the form dz on $\mathbb{C}$, when representing the torus as the quotient of $\mathbb{C}$ by a lattice. The translation structure from dz is just the usual flat metric from the plane with no singularities. It is well-known that geodesics on a flat torus satisfy

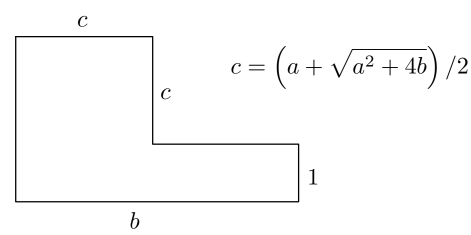

FIGURE 4.2. The L-shaped table $L(a, b)$.

the optimal dichotomy described in §2.2. Thus any billiard table which unfolds into a genus 1 translation surface must have optimal dynamics. In fact, using the genus formula (3.1), we see that there are only 4 such tables: the three triangles with angles $(\pi/3, \pi/3, \pi/3)$, $(\pi/2, \pi/4, \pi/4)$, $(\pi/2, \pi/3, \pi/6)$, and the square.

4.3. **Genus 2.** The story in genus 2 is significantly more complicated, and a complete discussion of billiard tables has involved a sophisticated understanding of the $\mathrm{SL}_2\mathbb{R}$ action on $\Omega^1\mathcal{M}_2$. McMullen found that translation surfaces in genus 2 for which $\Gamma(X, \omega)$ is *not* a lattice have geodesics as in Figure 2.2, showing:

Theorem 4.1. [Mc2] *If (X, ω) is a translation surface of genus 2 and $\Gamma(X, \omega)$ is not a lattice, then there exists a geodesic which is neither dense nor closed.*

For a different proof when ω has a double zero, see [Ca]. Consequently, we have:

Corollary 4.2. *If X has genus 2, then the following are equivalent:*

(1) *the translation surface (X, ω) is dynamically optimal;*
(2) *the translation surface (X, ω) satisfies the topological dichotomy; and*
(3) *the Veech group $\Gamma(X, \omega)$ is a lattice in $\mathrm{SL}_2\mathbb{R}$.*

McMullen went on to describe all Teichmüller curves and all dynamically optimal billiard tables in genus 2. To clarify the following statement, we need a few definitions. For any pair of integers a and b with $b > 0$, the billiard table $L(a, b)$ is shown in Figure 4.2. Two tables are equivalent if their unfolded surfaces lie in the same $\mathrm{SL}_2\mathbb{R}$ orbit.

Theorem 4.3. [Mc3] *Let T be a table which unfolds into a surface (X_T, ω_T) of genus 2. Then T is dynamically optimal if and only if it is equivalent to*

(1) *a table tiled by congruent triangles of angles $(\pi/2, \pi/3, \pi/6)$ or $(\pi/2, \pi/4, \pi/4)$;*
(2) *an L-shaped table $L(a, b)$ for some $a, b \in \mathbb{Z}$; or*
(3) *the triangle $(\pi/2, 2\pi/5, \pi/10)$.*

In fact, McMullen gave a complete description of the orbit-closures and invariant measures for the $\mathrm{SL}_2\mathbb{R}$ action on the moduli space $\Omega^1\mathcal{M}_2$ [Mc4].

5. The covering construction

In this final section, we discuss the basic idea which leads to Theorems 1.2 and 1.3. When a translation surface (Y, η) is a covering of another translation surface (X, ω), so that there is a covering map $p : Y \to X$ such that $\eta = p^*\omega$, then the Veech groups are commensurable [GJ]; that is, after conjugating in $SL_2\mathbb{R}$, the two groups share a finite-index subgroup. Gutkin and Judge used this relation of the Veech groups to characterize the translation surfaces (and billiard tables) with Veech groups commensurable to $SL_2\mathbb{Z}$. When two translation surfaces are related only by a *branched covering*, any holomorphic map of finite degree $Y \to X$ (so it may have critical points) which pulls ω back to η, then the Veech group structure is not necessarily preserved.

5.1. **The branched covers of Hubert-Schmidt.** Hubert and Schmidt considered branched covers of lattice surfaces (X, ω) branched over points of a special type. A point p in X is *periodic* if its orbit under the Veech group $\Gamma(X, \omega)$ is finite in X. (Note that the Veech group can be viewed as the group of diffeomorphisms from (X, ω) to itself which are linear with respect to the local Euclidean structure.) A point p is a *connection point* if any geodesic from a zero of ω through the point p again encounters a zero of ω. Hubert and Schmidt showed [HS]:

1. If p is a non-periodic connection point on a lattice surface (X, ω), then the subgroup $\Gamma(X, \omega, p) := \{\gamma \in \Gamma(X, \omega) : \gamma(p) = p\}$ is infinitely generated; and
2. If a branched cover $(Y, \eta) \to (X, \omega)$ over a lattice surface (X, ω) is branched only over a connection point p, then $\Gamma(Y, \eta)$ is commensurable with $\Gamma(X, \omega, p)$, and the surface (Y, η) satisfies the topological dichotomy.

They show further that the second statement holds when branching over more than one connection point if the base surface (X, ω) has a property they call *strong holonomy type.*

5.2. **Examples of Smillie-Weiss and Cheung-Hubert-Masur.** Smillie and Weiss made use of the Hubert-Schmidt construction to prove Theorem 1.2, concentrating on branched covers with a single ramification point in the base translation space of genus $g \geq 2$. They cleverly combine two facts: one is the simple observation that the forgetful map from $\mathcal{M}_{g,1}$ down to $\mathcal{M}_g$ has compact fibers. The second is Masur's theorem that a minimal non-uniquely ergodic direction gives rise to a particular $SL_2\mathbb{R}$-deformation which tends to infinity in $\Omega^1\mathcal{M}_g$ [Ma1].

In [CHM], the authors again use one of the Hubert-Schmidt branched covers, but branched over *two* points. The special example of Theorem 1.3 arises in the following way. Begin with the translation surface of the triangle with angles $(\pi/2, \pi/5, 3\pi/10)$. This triangle unfolds into two reflected copies of the regular pentagon, forming a surface of genus 2. It is dynamically optimal [Ve], and it is equivalent to one of the L-shaped tables in McMullen's classification Theorem 4.3 [Mc1, §9]. The centers of

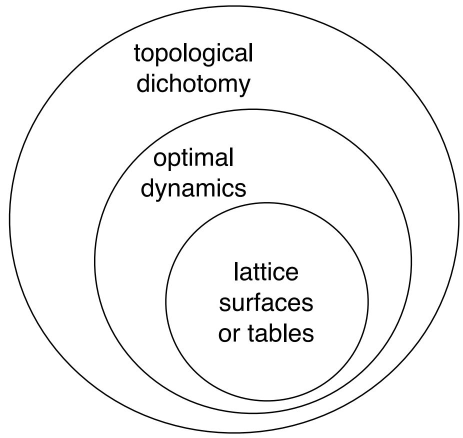

FIGURE 5.1. A diagram of inclusions for billiard tables or translation surfaces.

the two pentagons are connection points (see definition in §5.1). Taking a double cover of this genus 2 surface, branched over the two centers, produces the translation surface of genus 4 which is the unfolding of the triangle $(2\pi/5, 3\pi/10, 3\pi/10)$. The new surface satisfies the topological dichotomy by the arguments of [HS], but Cheung, Hubert, and Masur show that it has non-uniformly distributed, dense geodesics.

5.3. **Lattice tables and a dynamical characterization.** The schematic of Figure 5.1 indicates the relative inclusions of tables which have lattice Veech group, those which are dynamically optimal, and those satisfying topological dichotomy. By McMullen's theorem (Corollary 4.2), the sets coincide for translation surfaces of genus 2. The examples of [SW] and [CHM] show that the containments are strict in the setting of translation surfaces of arbitrary genus. Billiard tables have not yet been found which are dynamically optimal without lattice Veech group, but a search is under way. Further investigations are also in progress about possible characterizations of the lattice condition, because it is the geometry of the $\mathrm{SL}_2\mathbb{R}$ action on the moduli space of translation surfaces which drives most of the interest in this class of billiards.

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