

# AXIOM A POLYNOMIAL SKEW PRODUCTS OF $\mathbb{C}^2$ AND THEIR POSTCRITICAL SETS

-

## ERRATUM

LAURA DEMARCO AND SUZANNE LYNCH HRUSKA

### 1. INTRODUCTION

A polynomial skew product of  $\mathbb{C}^2$  is a map of the form

$$f(z, w) = (p(z), q(z, w)) = (p(z), q_z(w)),$$

where  $p$  and  $q$  are polynomials. We consider skew products which are Axiom A and extend holomorphically to endomorphisms of  $\mathbb{P}^2$  of degree  $d \geq 2$ . In the article [DH], we studied orbits of critical points in a distinguished subset of  $\mathbb{C}^2$ , and we constructed new examples of Axiom A maps.

We made an erroneous assumption about polynomial skew products, which holds for our main examples but fails in general. In this Correction, we describe the mistake and fix the proofs of our main results. We also indicate which statements in the original article do not hold without the extra assumption.

We would like to thank Hiroki Sumi for bringing the mistake to our attention and carefully describing an example for which the assumption fails (see [Su, Remark 4.13]). We also thank Shizuo Nakane for his careful reading of the original article.

**1.1. The extra assumption.** Let  $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$  be an Axiom A polynomial skew product. Let  $J_p \subset \mathbb{C}$  denote the Julia set of the base polynomial  $p$ . We define  $\Lambda \subset \mathbb{C}^2$  to be the subset of the nonwandering set of  $f$  contained in  $J_p \times \mathbb{C}$  and of saddle type. In particular,  $f|_\Lambda$  is expanding in the base direction (where it acts as the hyperbolic polynomial  $p$ ), and it is contracting in the vertical direction (acting by the polynomials  $q_z$  on fibers). The unstable manifold  $W^u(\Lambda)$  consists of all points  $x \in \mathbb{C}^2$  for which there exists a backward orbit  $x_{-k}$  of  $x$  converging to  $\Lambda$ . The stable manifold  $W^s(\Lambda)$  consists of all points  $x \in \mathbb{C}^2$  which converge to  $\Lambda$  under iteration.

The saddle set  $\Lambda$  decomposes into a disjoint union of *saddle basic sets*  $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_n$  on which  $f$  acts transitively and satisfies  $f(\Lambda_i) = \Lambda_i$ . For each saddle basic set, we have

$$W^s(\Lambda_i) \cap W^u(\Lambda_i) = \Lambda_i.$$

See [Jo2, Appendix A]. We incorrectly assumed

$$\text{Assumption } (*) \quad W^s(\Lambda) \cap W^u(\Lambda) = \Lambda$$

in many of our arguments.

Assumption (\*) holds when  $\Lambda$  is itself a saddle basic set. It also holds for products and for the main examples we give, as we explain below. It does not hold for Example 5.10 of the original article.

**1.2. Summary of the incorrect statements in the original article.** Lemma 3.5, Theorem 5.2, and Corollary 5.3 are false without Assumption (\*) and should be replaced with:

**Lemma E3.5.** *The unstable manifold of  $\Lambda$  satisfies*

$$W^u(\Lambda) \cap K_{J_p} = W^u(\Lambda) \cap W^s(\Lambda).$$

**Theorem E5.2.** *The following two conditions are equivalent:*

- (a)  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$ , and
- (b)  $z \mapsto \Lambda_z$  is continuous for all  $z \in J_p$ .

*These conditions imply Assumption (\*) and also*

- (c)  $z \mapsto K_z$  is continuous for all  $z \in J_p$ .

*Under Assumption (\*), condition (c) is equivalent to (a) and (b).*

**Corollary E5.3.** *If Assumption (\*) holds and  $J_z$  is connected for all  $z \in J_p$ , then  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$ .*

Claim (2) of Example 5.10 (which we added to the original example of Sumi) is false. In fact, Example 5.10 does not satisfy Assumption (\*) and provides a counterexample to the original versions of Lemma 3.5, Theorem 5.2, and Corollary 5.3.

Proposition 6.3 and Lemma 6.4 are proved under Assumption (\*); we do not have a proof of these statements in general and do not know if they are true.

**1.3. Correct statements in the original article requiring additional arguments for the proof.** Theorems 1.1, 5.1, and 6.1, and Propositions 1.2 and 5.4 of the original article are correct as stated, but the proofs there rely on Assumption (\*). In Section 2 we correct the proofs of the three theorems, and in Section 3 we correct the proofs of the two propositions.

## 2. CORRECTED PROOFS OF THE MAIN THEOREMS

In this section, we fix the proofs of Theorems 1.1, 5.1, and 6.1, which used Assumption (\*) in the original article, and we give the proof of the corrected Theorem E5.2.

**2.1. Proof of Theorem 1.1.** The proof of Theorem 1.1 relies on the lemmas of Section 3.

*Proof of Lemma E3.5.* The original proof of Lemma 3.5 provides a proof of the corrected version stated above in §1.2.

*Proof of Lemma 3.6.* The inclusion  $W^s(\Lambda) \subset K_{J_p} \setminus J_2$  follows from the definitions, since  $J_p \times \mathbb{C}$  is totally invariant, and  $p$  is expanding on  $J_p$ . For the reverse inclusion, suppose  $x \in K_{J_p} \setminus J_2$ . Its accumulation set  $A(x)$  lies in the nonwandering set of  $f$  within  $J_p \times \mathbb{C}$ . On the other hand, it is contained in  $W^u(\Lambda)$  by Proposition 3.3, so it is disjoint from  $J_2$ . Therefore  $A(x) \subset \Lambda$ , so  $x \in W^s(\Lambda)$ .  $\square$

*Proof of Lemma 3.7.* Let  $c$  be an element of the critical set  $C_{J_p}$ . Then the accumulation set  $A(c)$  is either empty or contained in the nonwandering set within  $K_{J_p}$ . From Proposition 3.3,  $A(c) \subset W^u(\Lambda)$ , so it is disjoint from  $J_2$ . Consequently  $A(c) \subset \Lambda$ , so  $A_{pt}(C_{J_p}) \subset \Lambda$ . The proof of the reverse inclusion is correct in the original article.  $\square$

**2.2. Proof of Theorem E5.2.** The corrected Theorem E5.2 is stated above in §1.2. The proofs that (a)  $\implies$  (b) and that (b)  $\implies$  (c) from the original article are correct. The proof that (c)  $\implies$  (a) is correct under Assumption (\*).

We first prove that (b)  $\implies$  (a). Suppose  $z \mapsto \Lambda_z$  is continuous over  $J_p$ , and fix  $x \in W^u(\Lambda)$ . Then there exists a prehistory  $x_{-k}$ , with  $f^k(x_{-k}) = x$ , converging to  $\Lambda$ . Continuity of  $\Lambda_z$  implies that a vertical neighborhood of  $\Lambda$  is in fact a neighborhood of  $\Lambda$  in the ambient space  $J_p \times \mathbb{C}$ . Consequently the points  $x_{-k}$  lie in a vertical trapping neighborhood of  $\Lambda$  for all large enough  $k$ . But then all forward iterates of these  $x_{-k}$  lie in smaller vertical trapping neighborhood, by Proposition 3.2, which implies that  $x$  is in  $\Lambda$ .

An easy argument shows that (a) implies Assumption (\*). Indeed, if  $x \in W^s(\Lambda) \cap W^u(\Lambda)$ , then Lemma 3.6 (with corrected proof given above in §2.1) implies that  $x$  lies in  $J_p \times \mathbb{C}$ . Then (a) implies that  $x \in \Lambda$ . The inclusion  $\Lambda \subset W^s(\Lambda) \cap W^u(\Lambda)$  is clearly true.  $\square$

**2.3. Proof of Theorem 5.1.** The proofs of (1) and (2) are correct as stated, so it remains to establish the validity of part (3). We apply Theorem E5.2, as stated above in §1.2 and proved in §2.2.

Assume that  $F_a$  is Axiom A, so that  $g_a$  is hyperbolic. Let  $P_a \subset \mathbb{C}$  denote the unique attracting cycle of  $g_a$ . Recall the notation for each fixed  $x \in \mathbb{C}$ ,

$$S_x = \{(e^{2it}, xe^{it}) \in \mathbb{C}^2 : t \in [0, 2\pi]\},$$

so that  $F_a(S_x) = S_{g_a(x)}$ . Define

$$\Lambda_a = \bigcup_{x \in P_a} S_x.$$

We will show that  $\Lambda = \Lambda_a$ . Observe first that  $F_a(\Lambda_a) = \Lambda_a$ . By computing derivatives of  $F_a$  along  $S_x$ , it is easy to see that saddle periodic points are dense in  $S_x$  for each  $x \in P_a$ . Therefore  $\Lambda_a \subset \Lambda$ . On the other hand, all points in  $K_{J_p} \setminus J_2$  converge to  $\Lambda_a$ , and since  $W^s(\Lambda) = K_{J_p} \setminus J_2$  by Lemma 3.6, we may conclude that  $\Lambda \subset \Lambda_a$ .

It follows immediately that  $z \mapsto \Lambda_z$  is continuous over  $J_p$  for these examples, and therefore by Theorem E5.2, we have  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$ .

**2.4. Proof of Theorem 6.1.** It remains to establish the validity of Theorem 6.1 parts (5) and (6).

At the end of the proof of Theorem 6.1, we applied Lemma 6.4 to conclude that  $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p})$ . The proof of Lemma 6.4 relies on Assumption (\*). Assumption (\*) holds for these examples because they have a unique saddle basic set. Alternatively, we can simply observe that  $A_{cc}(C_{J_p})$  must contain a unbounded connected set which intersects  $K_{J_p}$ , while  $A_{pt}(C_{J_p})$  is compact. This proves Theorem 6.1 (5).

To prove Theorem 6.1 (6), we used Proposition 6.3, which again relies on Assumption (\*). Alternatively, we can simply observe that the fiber Julia sets of these  $f_n$  are not all homeomorphic; so applying Corollary 4.4 we can conclude that  $f_n$  does not lie in the same hyperbolic component as a product.

### 3. $\hat{\Lambda}$ -STABILITY AND HYPERBOLIC COMPONENTS

In this final section, we complete the proofs of Propositions 1.2 and 5.4 of the original article. We show that the hypotheses imply Assumption (\*) and that the assumption holds throughout the hyperbolic component. We use the stability of the *natural extension* of the non-wandering set for Axiom A endomorphisms, as described in [Jo2, Appendix A]. Specifically, the natural extension of  $\Lambda$  moves holomorphically, in the sense of [Jo1, Theorem B], and the motion respects the skew-product structure.

**3.1. Skew motion of  $\hat{\Lambda}$ .** Jönsson showed that there is a motion of the natural extension of  $\Lambda$  on a neighborhood of  $f$  in the space of polynomial skew products. We begin by showing that this motion preserves the skew structure, exactly as in our Theorem 1.4 of the original article for the expanding part of the non-wandering set.

Let  $\{f_a = (p_a, q_a) : a \in \mathbb{D}\}$  be a holomorphic family of Axiom A skew products, and let  $\Lambda$  be the saddle set for  $f_0$  inside  $J_p \times \mathbb{C}$ . Combining [Jo1, Theorem B] and [Jo2, Corollary 8.14], there exist a radius  $r > 0$  and a continuous map

$$h : \mathbb{D}_r \times \hat{\Lambda} \rightarrow \mathbb{C}^2$$

satisfying

- (1) for each  $a \in \mathbb{D}_r$ ,  $\Lambda_a := h_a(\hat{\Lambda})$  is the saddle set of  $f_a$  in  $J_{p_a} \times \mathbb{C}$ ;
- (2) for each  $a \in \mathbb{D}_r$ ,  $h_a$  lifts to a conjugating homeomorphism  $\hat{h}_a : \hat{\Lambda} \rightarrow \hat{\Lambda}_a$ ; and
- (3)  $h(\cdot, \hat{x})$  is holomorphic for each fixed  $\hat{x}$  in  $\hat{\Lambda}$ .

Let  $\phi_a : J_p \rightarrow J_{p_a}$  be the conjugating homeomorphism for the holomorphic motion of the base Julia set  $J_p$ . We use density of periodic points in  $\Lambda$  to show that the

diagram

$$\begin{array}{ccc}
 \hat{\Lambda} & \xrightarrow{\hat{h}_a} & \hat{\Lambda}_a \\
 \hat{\pi} \downarrow & & \downarrow \hat{\pi} \\
 \hat{J}_p & \xrightarrow{\hat{\phi}_a} & \hat{J}_{p_a}
 \end{array}$$

commutes. Here  $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}$  is the projection to the first coordinate, so  $\hat{\pi}$  is the extension of this projection to the natural extension of the skew product.

Indeed, if  $\hat{x} \in \hat{\Lambda}$  is a periodic itinerary for  $f = f_0$ , then the first coordinate  $\hat{z} = \hat{\pi}(\hat{x})$  is periodic in  $\hat{J}_p$ . Under the extended motion  $\hat{h}_a$ , the itinerary  $\hat{x}_a$  remains periodic, so the projection  $\hat{z}_a = \hat{\pi}(\hat{x}_a)$  does too. By uniqueness of local solutions to  $p_a^n(z) = z$  near repelling periodic points, we conclude that  $\hat{z}_a = \hat{\phi}_a(\hat{z})$ , the image of  $\hat{z}$  under the extended motion of  $J_p$ . Consequently, the diagram commutes on periodic cycles. Periodic cycles are dense and the maps in the diagram are continuous; therefore the diagram commutes everywhere. In other words, the motion preserves the skew structure.

**3.2. The motion of  $\hat{\Lambda}$  induces a fiberwise continuity.** Fix a point  $z \in J_p$ , and let  $\hat{\Lambda}_z$  be the set of itineraries  $(x_{-k})$  in  $\hat{\Lambda}$  with initial point  $x_0$  in the fiber  $\{z\} \times \mathbb{C}$ . Then if  $f_a$  is a family of skew products as in §3.1, the skew-structure of the motion  $h$  implies that for each fixed  $a \in \mathbb{D}_r$ , the image  $h_a(\hat{\Lambda}_z)$  lies in the fiber  $\{z_a\} \times \mathbb{C}$ , with  $z_a = \phi_a(z)$ , and it coincides with the intersection  $(\Lambda_a)_{z_a} := \Lambda_a \cap (\{z_a\} \times \mathbb{C})$ . The continuity of  $h$  implies, and here is the key point, that the function

$$a \mapsto (\Lambda_a)_{z_a}$$

is continuous in the Hausdorff topology for each fixed  $z \in J_p$ .

**3.3. Discontinuity of  $z \mapsto \Lambda_z$  is an open condition.** Suppose  $z \mapsto \Lambda_z$  is discontinuous for an Axiom A skew product  $f$ , and let  $f_a$  be a holomorphic family with  $f_0 = f$  and for which there exists a holomorphic motion as in §3.1. Because  $\Lambda$  is closed, it must be that  $z \mapsto \Lambda_z$  fails to be *lower* semicontinuous at a point  $z_0 \in J_p$ . Thus, there is a point  $x = (z_0, w_0) \in \Lambda_{z_0}$ , a neighborhood  $U$  of  $w_0$  in  $\mathbb{C}$ , and a sequence of points  $z_k$  converging to  $z_0$  in  $J_p$  so that  $\Lambda_{z_k}$  does not intersect  $U$ . The continuity of  $(\Lambda_a)_{z_a}$  from §3.2 implies that under perturbation, the point  $x_a$  persists in  $\Lambda_a$  in the fiber over  $(z_0)_a$  while the neighborhood  $U$  remains empty in the fibers over moved points  $(z_k)_a$ . In other words,  $z \mapsto (\Lambda_a)_z$  is discontinuous also for nearby maps  $f_a$ .

While the discussion of holomorphic motions is done for 1-parameter families, defined over a disk  $\mathbb{D}$ , it should be remarked that the same arguments go through for higher-dimensional parameter spaces. Thus, the discontinuity of  $z \mapsto \Lambda_z$  is an open condition in the space of Axiom A skew products.

**3.4. Assumption (\*) holds on an open set.** The meaning of Assumption (\*) is that there are no relations among the basic sets of  $\Lambda$  in the preordering by intersection of stable and unstable manifolds. If Assumption (\*) is satisfied for a map  $f$ , then

the  $\hat{\Lambda}$ -stability of an Axiom A polynomial skew product guarantees that Assumption (\*) is satisfied on an open set containing  $f$ : no new relations can appear under perturbation [Jo2, Corollary 8.14].

**3.5. Proof of Proposition 5.4.** We aim to show that the condition  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$  holds throughout a hyperbolic component  $H$  if it holds for a single map. From the corrected Theorem E5.2, the equality  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$  implies Assumption (\*) and the continuity of  $z \mapsto K_z$ . The original proof of Proposition 5.4 shows that the continuity of  $z \mapsto K_z$  must hold throughout the entire hyperbolic component. Therefore, by the corrected Theorem E5.2, the desired equality  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$  is equivalent to Assumption (\*) in  $H$ .

Now, from §3.4, we know that Assumption (\*) holds on an open set in the hyperbolic component  $H$ . On the other hand, suppose there is a map  $f_1$  in the hyperbolic component for which  $z \mapsto \Lambda_z$  is discontinuous. The discontinuity of  $z \mapsto \Lambda_z$  is an open condition in  $H$  by §3.3, so we must have that Assumption (\*) fails on an open set in the hyperbolic component. This is a contradiction, so  $f_1$  cannot exist. Consequently, both Assumption (\*) and the equality  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$  hold throughout  $H$ .  $\square$

**3.6. Proof of Proposition 1.2.** We need to show that  $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$  for all maps in the hyperbolic component of a product. This is an immediate consequence of Proposition 5.4, because the condition holds for products. Indeed, it is immediate to see that  $z \mapsto \Lambda_z$  is continuous for products.  $\square$

## REFERENCES

- [DH] L. DeMarco and S. Hruska. Axiom A polynomial skew products of  $\mathbf{C}^2$  and their postcritical sets. *Ergodic Theory Dynam. Systems* **28**(2008), 1729–1748.
- [Jo1] M. Jonsson. Holomorphic motions of hyperbolic sets. *Michigan Math. J.* **45**(1998), 409–415.
- [Jo2] M. Jonsson. Dynamics of polynomial skew products on  $\mathbf{C}^2$ . *Math. Ann.* **314**(1999), 403–447.
- [Su] H. Sumi. Dynamics of postcritically bounded polynomial semigroups III: classification of semi-hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles. To appear, *Ergodic Theory Dynam. Systems*. Available at <http://www.math.sci.osaka-u.ac.jp/~sumi/>.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO  
*E-mail address:* demarco@math.uic.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF WISCONSIN MILWAUKEE  
*E-mail address:* shruska@uwm.edu