Math 535
Final Exam: some solutions
May 12, 2009

3. Construct a conformal isomorphism from the half-infinite strip

$$\{z = x + iy : x < 0 \text{ and } -\pi < y < \pi\}$$

to the upper half-plane $\{z = x + iy : y > 0\}$. Use compositions of exponentials, logs, powers, and/or linear fractional transformations, and draw pictures for each step. If you use $\sin z$ or $\cos z$, can you find a simpler formula?

> The function $f_1(z) = e^z$ sends the half-infinite strip conformally to a slit disk $S = \mathbb{D}\backslash(-1, 0]$. The principal branch of the square root $f_2(z) = z^{1/2}$ sends the slit disk S to the right half-disk $R = \{z \in \mathbb{D} : \operatorname{Re} z > 0\}$. The linear fractional transformation $f_3(z) = (-i)(z+i)/(z-i)$ sends R to the first quadrant $Q = \{z = x+iy : x, y > 0\}$. The squaring function $f_4(z) = z^2$ takes Q conformally to the upper half-plane. Our solution is
>
> $$f(z) = f_4(f_3(f_2(f_1(z)))) = \left(\frac{(-i)(e^{z/2} + i)}{e^{z/2} - i}\right)^2.$$
>
> On the other hand, the function $\sin z$ takes the half strip $\{z = x + iy : -\pi/2 < x < \pi/2, y > 0\}$ conformally to the upper half-plane (why?), so we could also use
>
> $$g(z) = \sin(-iz/2).$$

4. Let $f(z) = \tan z = \frac{\sin z}{\cos z}$, and let

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent expansion of f in the annulus $\{3 < |z| < 4\}$. Define

$$f_+(z) = \sum_{n=0}^{\infty} a_n z^n \qquad f_-(z) = \sum_{n=-\infty}^{-1} a_n z^n$$

(a) Prove that f is meromorphic on $\mathbb{C}$. What are the orders of its poles?

> The function f fails to be analytic only at the zeroes of $\cos z$. It is easy to check that these are all simple zeroes, since $\cos' z = -\sin z \neq 0$ where $\cos z = 0$. Hence the isolated singularities of f are simple poles.

(b) Compute a_{-1}, and obtain an explicit expression for f_-.

> The coefficient a_{-1} is the integral of f on a circle of radius between 3 and 4 (divided by $2\pi i$). By the residue theorem, this is the sum of the residues of f inside the circle. We need only compute the residue of f at $\pi/2$ and $-\pi/2$. We have
>
> $$\operatorname{Res}_{\pi/2}(f) = \lim_{z\to\pi/2}(z - \pi/2)f(z) = \frac{\sin(\pi/2)}{\cos'(\pi/2)} = -1.$$
>
> Similarly, $\operatorname{Res}_{-\pi/2}(f) = -1$, so $a_{-1} = -2$.

By uniqueness of the Laurent expansion, the function f_- is the "polar part" of f inside the disk of radius 3. Because f has only two simple poles in this disk, each with residue of -1, we see that

$$f_-(z) = \frac{-1}{z-\pi/2} + \frac{-1}{z+\pi/2}.$$

Recognizing each of these terms as the sum of a geometric series in $1/z$, we can expand f_- to find:

$$f_-(z) = -\sum_{n=1}^{\infty}\left(\frac{\pi}{2}\right)^{n-1}\frac{1}{z^n} - \sum_{n=1}^{\infty}\left(\frac{-\pi}{2}\right)^{n-1}\frac{1}{z^n} = -2\sum_{n \text{ odd}}\left(\frac{\pi}{2}\right)^{n-1}\frac{1}{z^n}.$$

(c) What is the radius of convergence of the power series f_+, and why?

The function $f_+ = f - f_-$ has simple poles at all zeroes of $\cos z$ except $\pm\pi/2$, where it has removable singularities. Thus the largest disk centered at 0 on which f_+ is analytic has radius $3\pi/2$. Consequently, the power series expansion for f_+ centered at 0 has radius of convergence $3\pi/2$.

5. Suppose f is a bounded analytic function on the upper half plane, $\mathrm{H} = \{z = x+iy : y > 0\}$, and $M > 0$ is a constant such that

$$\limsup_{z\to X}|f(z)| \le M$$

for each point $X \in \mathbb{R}$. Show that $|f(z)| \le M$ for all $z \in \mathrm{H}$.

Fix $\varepsilon > 0$ and define $g(z) = (z+i)^{-\varepsilon}f(z)$. Since $|z+i| > 1$ for all $z \in \mathrm{H}$, we have $|g(z)| \le |f(z)|$ for all $z \in \mathrm{H}$. As f is bounded, let $B > 0$ be chosen so that $|f(z)| \le B$ for all $z \in \mathrm{H}$. Since $|z+i|^{-\varepsilon} \to 0$ as $|z| \to \infty$, there exists $R > 0$ so that $|g(z)| \le |f(z)|\,M/B \le M$ for all $|z| \ge R$.

Fix $\varepsilon' > 0$. For each $X \in [-R, R]$, let U_X be a neighborhood of X so that $|f(z)| < M + \varepsilon'$ for all $z \in U_X \cap \mathrm{H}$. As $[-R, R]$ is compact, we may choose $\delta > 0$ so that the union $\bigcup_X U_X$ contains a δ-neighborhood of $[-R, R]$. Let Γ_δ be the boundary of the upper half-disk of radius R centered at $z_0 = \delta i$, oriented positively. Then for all z in the "horizontal part" of Γ_δ, namely $\{z \in \Gamma_\delta : \operatorname{Im} z = \delta\}$, we have $|g(z)| \le |f(z)| < M + \varepsilon'$. On the rest of Γ_δ, we have $|z| \ge R$, so $|g(z)| \le M$. By the Maximum Principle, we find that $|g(z)| \le M + \varepsilon'$ for all z inside Γ_δ.

Now observe that for each fixed $z_0 \in \mathrm{H}$, we have $\lim_{\varepsilon\to 0} g(z_0) = f(z_0)$. Choose R_0 large enough and δ_0 small enough that z_0 lies inside Γ_δ for all $R \ge R_0$ and all $\delta \le \delta_0$. By choosing an R and then an appropriate δ, we find that $|g(z_0)| \le M + \varepsilon'$. This estimate holds for any choice of ε' and ε, so we may conclude that $|f(z_0)| \le M$.

6. Let $U(\varepsilon) \subset \mathbb{C}$ be the ε-neighborhood of the real interval $[0, 1]$. That is, $U(\varepsilon)$ is the set of all points $z \in \mathbb{C}$ such that $|z - x| < \varepsilon$ for some $0 \le x \le 1$. Fix point $p = 1/2$, and let f_ε be the Riemann map from $U(\varepsilon)$ to the disk $\mathbb{D}$, with $f_\varepsilon(p) = 0$ and $f'_\varepsilon(p) > 0$. Prove that

$$f'_\varepsilon(p) \to \infty$$

as $\varepsilon \to 0$.

Consider the family $\mathcal{F}$ of inverse functions f_ε^{-1}, for all $0 < \varepsilon < 1$. Each $f \in \mathcal{F}$ is analytic on the unit disk with image contained in the neighborhood $U(1)$; consequently the family is uniformly bounded. By Montel's Theorem, $\mathcal{F}$ is a normal family. For any sequence $\varepsilon_n \to 0$, there is a subsequence $\varepsilon_{n_k} \to 0$ such that $f_{\varepsilon_{n_k}}^{-1}$ converges uniformly on compact subsets. Let g be its limit. As $f_\varepsilon(p) = 0$ for all ε, we see that $g(0) = p$. By Weierstrass' Theorem, g is analytic. If g is non-constant, then by the Open Mapping Theorem, its image is open and contains a small disk D around the point $p = 1/2$. This contradicts the construction of g. Indeed, fix an open subset V in D which does not intersect $U(\delta)$ for some $\delta > 0$. Then for all $\varepsilon < \delta$, the image of f_ε^{-1} is $U(\varepsilon) \subset U(\delta)$ which is disjoint from V. It follows that the image of g cannot intersect V. We conclude that g must be constant. As $g(0) = p$, we have $g(z) = p$ for all z. The limit is independent of the sequence ε_n, so the family f_ε^{-1} converges to g as $\varepsilon \to 0$, uniformly on compact subsets of $\mathbb{D}$.

As g is constant, its derivative is constant 0, so by Weierstrass' Theorem again, the derivative of f_ε^{-1} converges uniformly to 0 on compact subsets. In particular, $(f_\varepsilon^{-1})'(0) \to 0$, so

$$f'_\varepsilon(p) = \frac{1}{(f_\varepsilon^{-1})'(0)} \to \infty$$

by the Inverse Function Theorem.

Remark. Even though the sets $U(\varepsilon)$ nest down to the interval $[0,1]$, the functions f_ε^{-1} converge to a constant function, and *not* to a function with image $[0,1]$.

7. Suppose Ω is a simply connected region in $\mathbb{C}$ which is symmetric about the real axis. That is, $z \in \Omega$ if and only if $\bar{z} \in \Omega$. Fix a point $p \in \Omega \cap \mathbb{R}$. Prove that the Riemann map $f : \Omega \to \mathbb{D}$, with $f(p) = 0$ and $f'(p) > 0$, satisfies

$$f(\bar{z}) = \overline{f(z)}.$$

Conclude, in particular, that $f(x) \in \mathbb{R}$ for all $x \in \Omega \cap \mathbb{R}$.

Consider the function $g(z) = \overline{f(\bar{z})}$. By checking the Cauchy-Riemann equations, we see that g is analytic on the domain $\{z \in \mathbb{C} : \bar{z} \in \Omega\} = \Omega$. As f is injective and surjective, we see that g is injective and surjective. Also, $g(p) = \overline{f(\bar{p})} = \overline{f(p)} = 0$. Writing $f = u + iv$, we also have

$$g'(p) = \frac{\partial u}{\partial x}(p) - i\frac{\partial v}{\partial x}(p) = \overline{f'(p)} = f'(p) > 0,$$

so g is a Riemann map. By uniqueness, we conclude that $g = f$. In other words,

$$f(\bar{z}) = \overline{f(z)}.$$

For $x \in \Omega \cap \mathbb{R}$, we have $x = \bar{x}$, so $f(x) = \overline{f(x)}$ implies $f(x)$ is real.

8. (a) Give an example of a region $\Omega \subset \mathbb{C}$ with a trivial group of conformal automorphisms. That is, find an Ω so that if $f : \Omega \to \Omega$ is analytic and invertible, then f is the identity. Justify your answer.

Try the punctured disk $\Omega = \mathbb{D} \setminus \{0, 1/4, 1/2\}$. Any automorphism of Ω extends to an automorphism of $\mathbb{D}$, because the singularities are removable. The only automorphisms of $\mathbb{D}$ which take 0 into the set $\{0, 1/4, 1/2\}$ are rotations about 0 or rotations followed by an automorphism of the form

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

where a is $1/4$ or $1/2$. (Why?) The only rotation preserving $\{0, 1/4, 1/2\}$ is the identity. Note that $\varphi_{1/4}(-1/4) = 1/3$. As a linear fractional transformation, $\varphi_{1/4}$ must send the circle $\{|z| = 1/4\}$ to a circle intersecting $\mathbb{R}$ perpendicularly at 0 and $1/3$. Therefore, any rotation followed by $\varphi_{1/4}$ sends $1/4$ into this circle; only the identity rotation can take $1/4$ into the set $\{0, 1/4, 1/2\}$. But $\varphi_{1/4}(1/2) = -2/7$, so it does not permute the set $\{0, 1/4, 1/2\}$. Similary, rotations followed by $\varphi_{1/2}$ do not permute the set $\{0, 1/4, 1/2\}$. Therefore, the conformal automorphism group of Ω is trivial.