Math 535
Homework 2
Due Friday, January 30

Read Chapter 2. You are encouraged to work on *all* of the exercises in the text, but you only need to turn in the following problems.

1. Let a, b, c, d be real numbers. Determine the conditions for which the polynomial

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$$

is harmonic. Find a harmonic conjugate v by two methods: integration and the formalism described on page 27. Express the analytic function $u + iv$ as a function of a complex variable z.

2. Let f be an analytic function such that $|f(z)|$ is constant. Show that f must be constant.

3. If Q is a polynomial with distinct roots $q_1, q_2, \ldots, q_n$, and if P is a polynomial of degree $< n$, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(q_k)}{Q'(q_k)(z - q_k)}.$$

4. Use the formula of the previous exercise to prove there exists a unique polynomial P of degree $< n$ with given values $P(q_k) = c_k$. This is called *Lagrange interpolation.*

5. Let $f(z) = P(z)/Q(z)$ be a rational function, and let

$$R(z) = \frac{Q(z)P'(z) - P(z)Q'(z)}{(Q(z))^2}$$

be the rational function which computes the derivative of f where $Q \neq 0$. If f has degree d, how large and how small can the degree of R be?

6. (a) Let p be an integer. Find the radius of convergence for the power series:

$$\sum_n n^p\, z^n, \ \sum_n n!\, z^n, \ \sum_n \frac{1}{n!}\, z^n, \ \sum_n z^{n!}$$

(b) If $\sum a_n z^n$ has radius of convergence R, what are the radii of convergence of $\sum a_n z^{2n}$ and $\sum a_n^2 z^n$?