

**Math 535**  
**Homework 3**  
Due Friday, February 13

Read Chapter 3. You are encouraged to work on *all* of the exercises in the text, but you only need to turn in the following problems.

1. Prove the Fundamental Theorem of Algebra, following this outline. Be sure to justify every step.

If  $p(z)$  is a nonconstant polynomial with no zeroes in  $\mathbf{C}$ , then  $|p(z)|$  attains its minimum at some point  $z_0$ . We can assume  $z_0 = 0$  and that  $p(z) = 1 + az^m + \cdots$ , where  $m \geq 0$  and  $a \neq 0$ . But then there exists  $\varepsilon > 0$  and  $\theta \in \mathbf{R}$  so that  $|p(\varepsilon e^{i\theta})| < 1$ , contradicting minimality at 0.

2. (a) For what values of  $z$  is

$$\sum_{n=0}^{\infty} \left( \frac{z}{1+z} \right)^n$$

convergent?

(b) For what values of  $z$  is

$$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}$$

convergent?

3. Determine the real and imaginary parts of  $z^z$ . Compute all possible values for  $z = -i$  and  $z = 1/4$ .

4. We can define the number  $\pi$  as the smallest positive period of  $e^{2iz}$ ; see Ahlfors §3.3. Use the exponential function to define *angles* of a triangle, bearing in mind that they should lie between 0 and  $\pi$ . With this definition, prove that the sum of the angles is  $\pi$ .

5. Suppose that  $f$  is analytic and satisfies

$$|f(z)^2 - 1| < 1$$

for all  $z$  in a region  $W$ . Show that either  $\operatorname{Re} f(z) > 0$  or  $\operatorname{Re} f(z) < 0$  for all  $z$  in  $W$ .

6. Show there is a well-defined branch of  $f(z) = \sqrt{z(1-z)}$  on  $W = \mathbf{C} \setminus [0, 1]$  which has positive imaginary part at  $z = 2$ . What is the image  $f(W)$ ? Describe the action of  $f$ , perhaps by sketching the image of some curves. What is  $f'(z)$  (and specify the branch); is  $f$  conformal; is  $f$  one-to-one?

7. (a) Show that any linear fractional transformation which sends the real axis (union  $\infty$ ) to itself can be written with real coefficients.

(b) Find the linear fractional transformation which carries the circle  $\{|z| = 2\}$  to  $\{|z+1| = 1\}$ , the point  $-2$  to 0, and 0 to  $i$ .