Math 535
Homework 7
Due Friday, March 20

Most of the following exercises use the ideas from Ahlfors Chapter 4, Section 3.4. Let $\mathbb{D}$ denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Recall that for any $a \in \mathbb{D}$, the linear fractional transformation

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

takes the unit disk conformally and homeomorphically to itself.

1. Suppose f is an analytic function on $\mathbb{D}$ such that $|f(z)| < 1$ for all z. Show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}$$

for all $z \in \mathbb{D}$. Show further that if equality holds for some $z_0 \in \mathbb{D}$, then f is a linear fractional transformation.

2. Suppose f is a conformal automorphism of the unit disk, so that f sends $\mathbb{D}$ conformally and homeomorphically to itself. Prove that

$$f(z) = e^{i\theta}\varphi_a(z)$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

3. A *finite Blaschke product* is a rational function of the form

$$B(z) = e^{i\theta} \prod_{j=1}^{n} \varphi_{a_j}(z)$$

for $\theta \in \mathbb{R}$ and points $a_1, \ldots, a_n \in \mathbb{D}$. By the previous exercise, the Blaschke products of degree 1 are precisely the conformal automorphisms of the disk. Show that if $f : \mathbb{D} \to \mathbb{D}$ is an analytic function which extends continuously to the boundary, so that $|f(z)| = 1$ for all $|z| = 1$, then f must be a finite Blaschke product.

4. Suppose f is analytic on the disk $\mathbb{D}_3 = \{z \in \mathbb{C} : |z| < 3\}$ with $|f(z)| < 1$ for all z. Suppose also that $f(i) = f(-i) = f(1) = f(-1) = 0$. What is the maximum value of $|f(0)|$? For which functions is the maximum attained?

5. Prove that any conformal automorphism of $\mathbb{C}$ is an affine transformation $A(z) = az + b$, for $b \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$. Hint: show first that the isolated singularity at infinity must be a pole of order 1.

6. Show that the conformal automorphisms of the punctured disk $\mathbb{D} \setminus \{0\}$ are the rotations $z \mapsto e^{i\theta}z$.

7. Show that any analytic function $f : \mathbb{D} \to \mathbb{D}$ which is not the identity has at most one fixed point.