Math 535
Homework 9
Due Monday, April 20

Read Chapter 5 §5 and Chapter 6 §1. You are encouraged to work on *all* of the exercises in the text, but you only need to turn in the following problems.

1. Suppose $\mathcal{F}$ is a family of analytic functions on a region $W \subset \mathbb{C}$ such that $\operatorname{Re} f > 0$ for all $f \in \mathcal{F}$. Fix a point $p \in W$. Prove that $\mathcal{F}$ is a normal family if $\sup_{f \in \mathcal{F}} |f(p)| < \infty$. Hint: consider e^{-f} and think carefully!

What happens if we eliminate the condition $\sup_{\mathcal{F}} |f(p)| < \infty$? See Ahlfors §5.5. Is the family normal according to Definition 3 of that section? Use this definition also for the following exercise.

2. If f is analytic on all of $\mathbb{C}$, show that the family of functions $\mathcal{F} = \{f(\alpha z) : \alpha \in \mathbb{C}\}$ is normal on the annulus

$$A = \{1 < |z| < 10\}$$

if and only if f is a polynomial (where in the definition of normal, we allow uniform convergence on compact subsets to the constant ∞).

3. Let B be a bounded region in $\mathbb{C}$ and b a point in B. Suppose $\{f_n : B \to B\}$ is a sequence of analytic functions such that for each derivative m, we have $f_n^{(m)}(b) \to 0$ as $n \to \infty$. Prove that f_n converges to the zero function uniformly on compact subsets of B. Is the result true for all regions B in $\mathbb{C}$?

4. Prove the following generalization of the Schwarz Lemma using normal families. Suppose $f : B \to B$ is an analytic function from a bounded region B to itself. If $f(b) = b$ for some $b \in B$, prove that $|f'(b)| \leq 1$. Hint: consider the sequence of compositions $f^n = f \circ f \circ \cdots \circ f$. Is the result true for unbounded regions B?

5. Suppose f is a bounded analytic function on the strip $\{x + iy \in \mathbb{C} : -1 < y < 1\}$, and suppose $f(x + i0) \to 0$ as $x \to +\infty$. Prove that for each $r < 1$, $f(x + iy) \to 0$ as $x \to +\infty$ uniformly for $y \in [-r, r]$.