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# The Dynamics of Newton's Method

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We consider the dynamics of a special class of rational functions — those rational functions that are obtained from Newton's method as applied to a polynomial equation. Such maps are interesting for two reasons:
(1) They form a natural family of non-polynomial examples; and
(2) their dynamical properties are related to their utility as numerical algorithms.

After reviewing a few basic facts, we describe a one-parameter family of degree-three rational functions derived from Newton's method applied to cubic equations in one variable. Then, in order to explain the results of a related computer experiment, we present Douady and Hubbard's remarkable theory of polynomial-like mappings.

Using Newton's method to find the roots of a polynomial equation

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \ldots + a_0 = 0$$

is identical to computing individual orbits of the dynamical system generated by the Newton map $N(z)$ (or $N_p(z)$ if we need to be explicit about the polynomial $p(z)$)

$$N(z) = z - \frac{p(z)}{p'(z)}. \tag{1}$$

As we discuss Newton's method, it is useful to keep the following somewhat typical example in mind.

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**Example 1.** If we apply Newton's method to the polynomial equation $p(z) = z^3 - 1 = 0$, we obtain the rational map

$$N(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}.$$

The three solutions to $z^3 = 1$ are 1 and $(-1 \pm i\sqrt{3})/2$. One can easily verify that this rational map has superattracting fixed points of multiplicity two at each of the roots. Since $\deg(N) = 3$, each root has a "third" inverse image somewhere in $\mathbb{C}$. In this example, these "prefixed" points correspond to distinct components of the Fatou set. They map injectively onto the immediate basins of the superattracting fixed points, i.e., the components of the Fatou set that contain the roots.

Note that, unlike a polynomial function, the inverse image of $\infty$ in this example contains more than one point. That is, $N^{-1}(\infty) = \{0, \infty\}$. In fact, as Figure 1 suggests, the preorbit of $\infty$ (all points that eventually map to $\infty$ under iteration) is dense in the Julia set. ◇

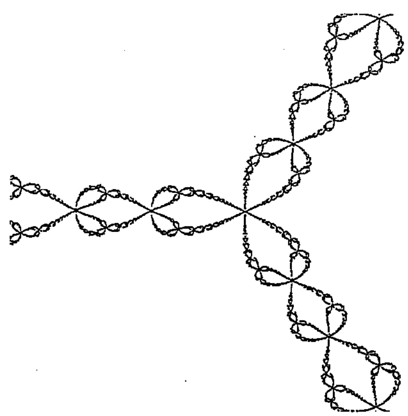


Figure 1. The Julia set of Newton's method applied to the equation $z^3 = 1$.

We begin our general discussion by summarizing a few basic facts about Newton's method for polynomials.

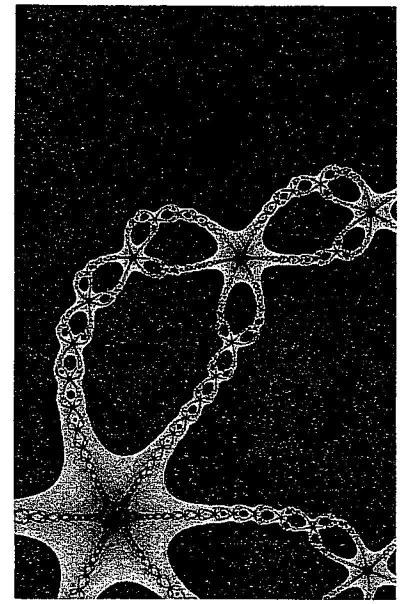
**Remark 1.**

a) From Equation 1, we see that the roots of $p(z)$ correspond to the finite fixed points of $N(z)$.
b) The point at infinity is a fixed point, and since $N'(\infty) = d/(d-1)$, it is repelling. Therefore, if Newton's method produces an iterate near $\infty$, successive iterates tend away from infinity.
c) The derivative of $N$ is

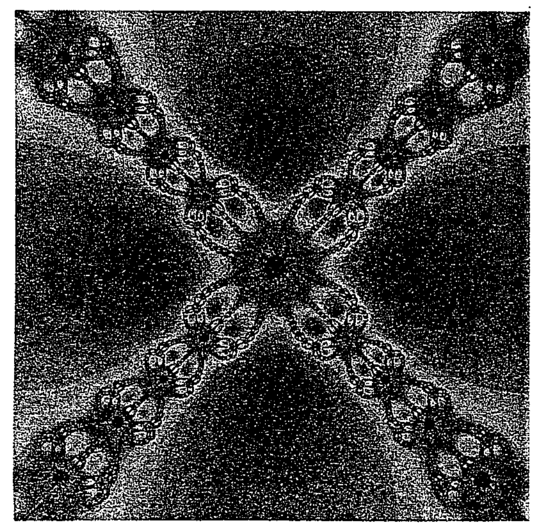
$$N'(z) = \frac{p(z)p''(z)}{[p'(z)]^2},$$

and therefore, the simple roots of $p(z)$ are superattracting fixed points of $N(z)$. This is a desirable property for a root-finding algorithm because, in a neighborhood of its superattracting fixed points, the algorithm is locally conjugate

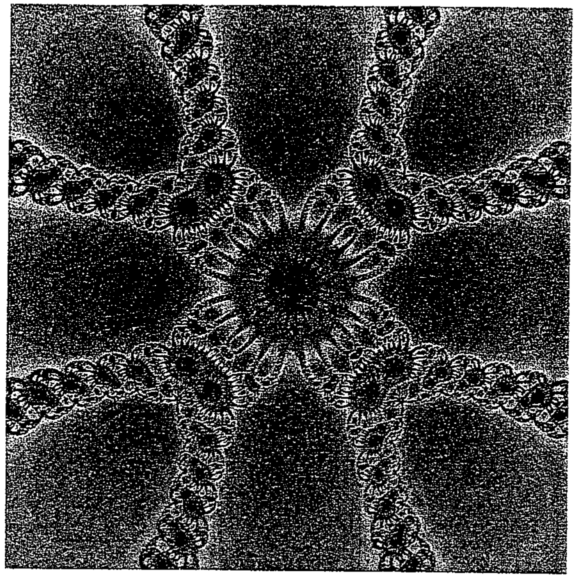




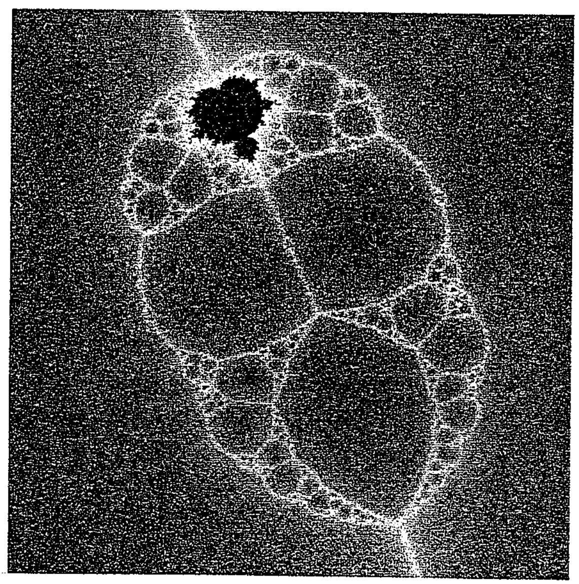
COLOR PLATE 1. An enlargement of part of the basin portrait for Newton's method applied to the equation $z^3 = 1$. The coloring scheme highlights the Julia set.



COLOR PLATE 2. The basin portrait for Newton's method applied to the equation $z^4 = 1$.



Color Plate 3. The basin portrait for Newton's method applied to the equation $z^8 + 3z^4 = 4$.



Color Plate 4. An enlargement of part of the parameter space for Newton's method applied to cubic polynomials. See Figure 6 on p. 146.



to $z \mapsto z^k$ for some $k > 1$. Thus, local convergence is very rapid. In fact, the number of decimal places of accuracy at least doubles with each iteration.

(d) Multiple roots are attracting fixed points, but they are not superattracting. In fact, for a multiple root of multiplicity $m$, the derivative of Newton's method at the root is $(m-1)/m$. Thus, the rate of attraction is linear, and the algorithm is not very effective in this case.

(e) For a generic polynomial of degree $d$, the Newton's map is a rational map of degree $d$. However, when the polynomial has multiple roots, $\deg(N) < d$. In this section, we mostly consider polynomials that do not have multiple roots.

(f) In other chapters of this volume, we have seen that dynamical properties of a complex-analytic function are often determined by the dynamics of its critical points. Therefore, it is important to note that the critical points of $N(z)$ are the simple roots as well as the inflection points of $p(z)$.

(g) Likewise, since $\infty$ is a (slowly) repelling fixed point for Newton's method, we note that the poles of $N(z)$ are the critical points of $p(z)$. Consequently, orbits that avoid the critical points of $p(z)$ have the best chance of converging rapidly to a root.

(h) Unfortunately, given (f) and (g), we must keep two sets of critical points in mind. Note that it is the critical points of $N(z)$ (a subset of the roots and inflection points of $p(z)$) that predict the overall dynamics of $N(z)$.

Given Remark 1(g), one wonders how the critical points of $p(z)$ are related to its roots.

**Theorem 1.** (Lucas, 1874) *The critical points of $p(z)$ are contained in the convex hull of the roots of $p(z)$.* ∎

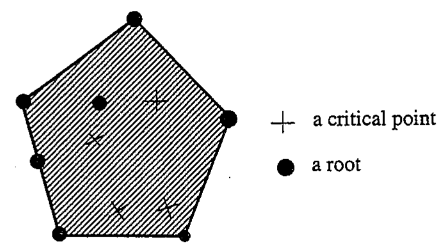



Figure 2. The critical points of $p(z)$ (i.e., the poles of $N(z)$) are contained in the convex hull of the roots of $p(z)$.

It is also useful to consider the manner in which Newton's method for a given polynomial is related to Newton's method for a "rescaled" polynomial.

**Remark 2.** Let $T(z) = \alpha z + \beta$ where $\alpha \neq 0$ and let $q(z) = p(T(z))$. Then

$$T \circ N_q \circ T^{-1} = N_p.$$

In other words, we can transform the roots by an affine map without qualitatively changing the dynamics of the corresponding Newton's function.

Given these general observations, we are now ready to discuss global convergence properties. Due to its simplicity, we start with the quadratic case. The result we describe has been known since at least 1870 and is explicitly discussed in the papers of Schoeder [Sch] and Cayley [C1–4] (see [A] for an interesting discussion of the history of this result).

**Theorem 2.** *Let $p(z)$ be a quadratic with distinct roots. Then Newton's method $N_p(z)$ is globally, analytically conjugate to the quadratic polynomial $z \mapsto z^2$.*

*Proof.* We can establish this result without doing any calculation. Denote the roots of the quadratic by $\alpha$ and $\beta$ and consider the Möbius transformation

$$h(z) = \frac{z-\beta}{z-\alpha}.$$

Note that $h(\infty) = 1$, $h(\beta) = 0$, and $h(\alpha) = \infty$. Then, $h \circ N_p \circ h^{-1}$ is a rational map of degree 2 that has superattracting fixed points at 0 and $\infty$, and it fixes 1. It is therefore the map $z \mapsto z^2$. ∎

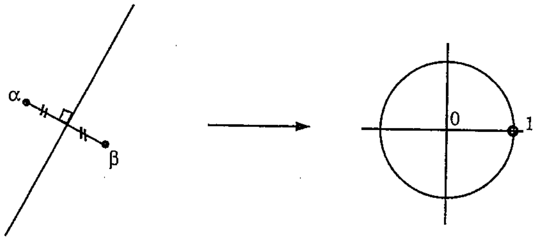



Figure 3. The conjugacy $h$ in Theorem 2.

Note that, under the conjugacy $h$, the Julia set for $z \mapsto z^2$ corresponds to the perpendicular bisector of the line segment from $\alpha$ to $\beta$. Along this bisector, $N_p$ has the "angle doubling" dynamics of the map $z \mapsto z^2$ restricted to the unit circle.

As we mentioned above, the critical points for Newton's method are the roots and the inflection points of the polynomial. Since the roots are always fixed, the only "free" critical points are the inflection points. In the quadratic case, there are no inflection points, and Newton's method is always conjugate to the rational map $z \mapsto z^2$.

The analysis of Newton's method becomes dramatically more complicated as soon as the degree of the polynomial equation is greater than 2. To see why, we describe work done in the early 1980's by Curry, Garnett, and Sullivan [CGS] and by Douady and Hubbard [DH]. (Tan Lei [L] has recently completed a detailed mathematical description of this parameter space.)

**A Computer Experiment:**

To study Newton's method for cubics using computer graphics, we use the observation in Remark 2 to eliminate duplication. Given three distinct roots in $\mathbb{C}$, there is an affine map that transforms this "triangular" configuration to one that is entirely located in the half plane $\{z \mid \operatorname{Im} z \geq 0\}$ with its longest side being the interval $[0, 1]$. Therefore, we need only consider polynomials of the form

$$p_\rho(z) = z(z-1)(z-\rho), \qquad (2)$$

where $\operatorname{Im} \rho \geq 0$, $|\rho| \leq 1$, and $|\rho - 1| \leq 1$. In other words, we use a one-parameter family of polynomials whose roots are 0, 1, and $\rho$, where $\rho$ is the parameter (see



Figure 4). Moreover, by identifying the triangles in Figure 4 that are congruent via a conformal affine map, we see that this parameter space is homeomorphic to the two-sphere $S^2$ with one puncture. (In our computer pictures we let $\rho$ range throughout $\mathbb{C}$.)

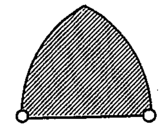


Figure 4. This figure is a sketch of the region in the $\rho$-plane that corresponds to a completely representative collection of cubic polynomials in the form of Equation 2. Given an arbitrary cubic $q$, there is a number $\rho$ in this region such that $N_q$ and $N_{p_\rho}$ are globally conjugate by Remark 2.

Given this representative family of cubics, we now explore the dynamics of the associated Newton functions. In general, given a polynomial $p(z)$, the Julia set of $N_p$ does not play an important role for two reasons. In all known examples, it has measure zero, and it usually repells nearby points. Therefore, numerical errors force most orbits to move away from the Julia set.

Consequently, we focus on the structure of the Fatou set $F$. Computer evidence suggests that the basins of the roots are relatively large components of $F$. However, $F$ may contain basins of other attracting periodic orbits. If such orbits exist, they do not correspond to roots of $p(z)$, and every starting point in such a basin leads to an unsuccessful application of Newton's method. The following experiment ([CGS] and [DH]) is designed to locate such orbits.

Recall that all periodic attracting orbits attract at least one critical point. Thus, we use critical points to locate attracting periodic orbits. We follow the orbit of the "free" critical point of $N_p$, the inflection point of $p(z)$. In Figures 5 – 7, we shade the parameter value $\rho$ according to the number of iterates it takes for the orbit of the associated inflection point to converge to one of the three roots. If it does not converge, we plot a black point at $\rho$.

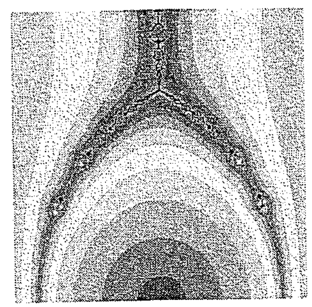


Figure 5. This figure contains the unit square

$$\{z \mid 0 \leq \operatorname{Re} z \leq 1,\ 0 \leq \operatorname{Im} z \leq 1\}$$

in the $\rho$-plane. Thus it contains the region sketched in Figure 4, and it is shaded using the scheme described directly above.

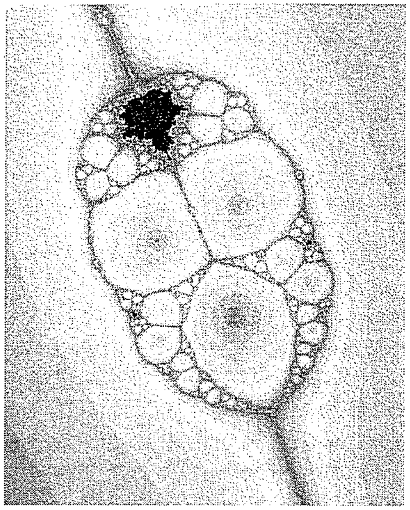


Figure 6. This figure is an enlargement of a small rectangle from Figure 5. Precise coordinates are given in the appendix. Recall that the color black corresponds to parameters for which the orbit of the inflection point does not converge to one of the three roots.

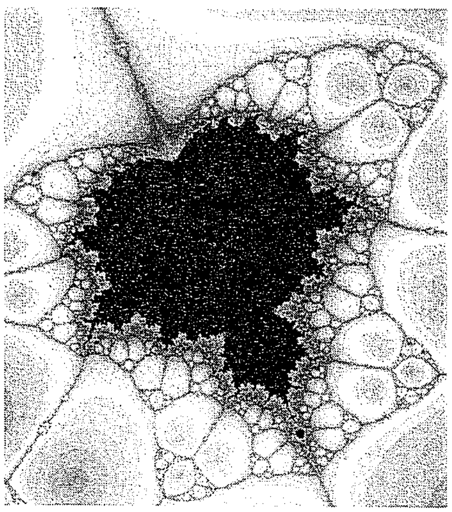


Figure 7. This figure is an enlargement of a rectangle from Figure 6. It illustrates the largest of black regions in Figure 6.

Figures 5 – 7 describe portions of parameter space. We use these figures to determine interesting values of the parameter $\rho$. The following two figures illustrate the structure of the Fatou set for a value of $\rho$ chosen from the black region in Figure 7.



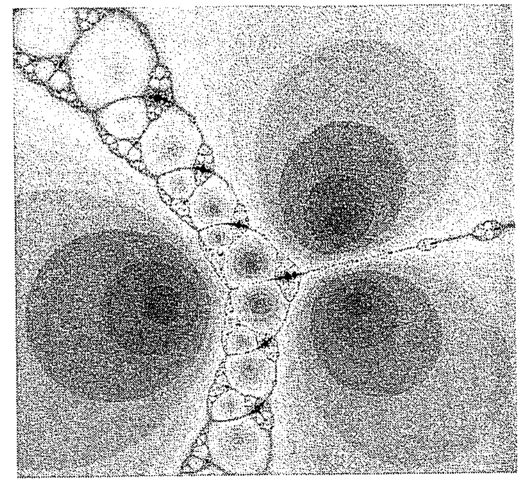


Figure 8. This picture illustrates the structure of the Fatou set for $\rho = 0.909419 + 0.416106i$. We shade a point corresponding to the number of iterates necessary for its orbit to converge to one of the roots (up to a reasonable accuracy). If the orbit does not converge after a prescribed number of iterates, a black dot is plotted at the initial point.

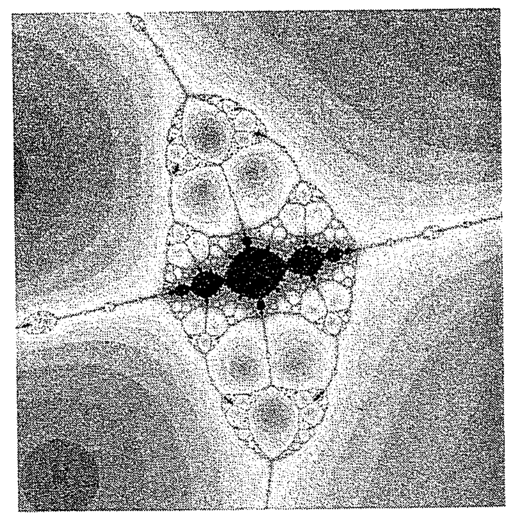


Figure 9. This figure is an enlargement of a rectangle containing one of the black regions in Figure 8. The large black region $K$ that resembles the filled Julia set of $z \mapsto z^2 - 1$ is invariant under the second iterate of $N$. The union $K \cup N(K)$ contains a superattracting periodic orbit whose period is 4.

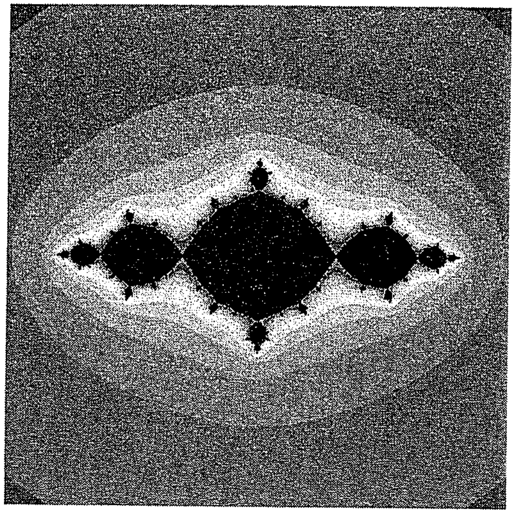


Figure 10. The filled Julia set of $z \mapsto z^2 - 1$. We include this figure to emphasize the similarity between Figure 9 and this filled Julia set.

The results of this experiment are remarkable. We are studying the dynamics of maps that bear little relationship with quadratics, and we are considering a parameter space that presumably has nothing to do with the quadratic family. Why do we obtain images of the Mandelbrot set? What is their significance?

Douady and Hubbard [DH] answered both questions when they developed their theory of polynomial-like mappings.

**Definition.** Suppose that $U'$ and $U$ are simply-connected domains and that $U'$ is a relatively compact subset of $U$. A map $f\colon U' \to U$ that is analytic and proper is called a *polynomial-like* map.

**Remarks:**

(a) A map $f$ is *proper* if the inverse image of a compact set is compact.

(b) A polynomial-like map has a finite degree which can be determined by counting inverse images with multiplicity.

(c) The definition of a polynomial-like map depends both on the analytic map and the choice of $U'$. For example, it is possible to find cubics that are, of course, cubic-like on disks of large radius around the origin but that are quadratic-like on smaller domains. Also, transcendental functions can be polynomial-like if the domain $U'$ is chosen appropriately. See Examples 2 and 3.

**Example 2.** Given a cubic polynomial $p(z)$, we define a function $h_p : \overline{\mathbb{C}} \to [0, \infty]$ that is harmonic on the basin of $\infty$ by

$$h_p(z) = \lim_{k \to \infty} \tfrac{1}{3^k} \log_+ |p^k(z)|$$

where $\log_+ = \max\{0, \log\}$. In general, $p(z)$ has two distinct critical points $c_1$ and $c_2$. For any cubic $p(z)$ that has $c_1 \in W^s(\infty)$ and $c_2 \notin W^s(\infty)$, we obtain a a polynomial-like map $q$ of degree 2 by restricting $p(z)$. Let $v = h(c_1)$. Then $h(p(c_1)) = 3v$. We set $U = \overline{\mathbb{C}} - h^{-1}[3v, \infty]$ and choose $U'$ to be the component of $p^{-1}(U)$ that contains $c_2$ (see Figure 11). The map $p : U' \to U$ is a polynomial-like map of degree 2. ◇



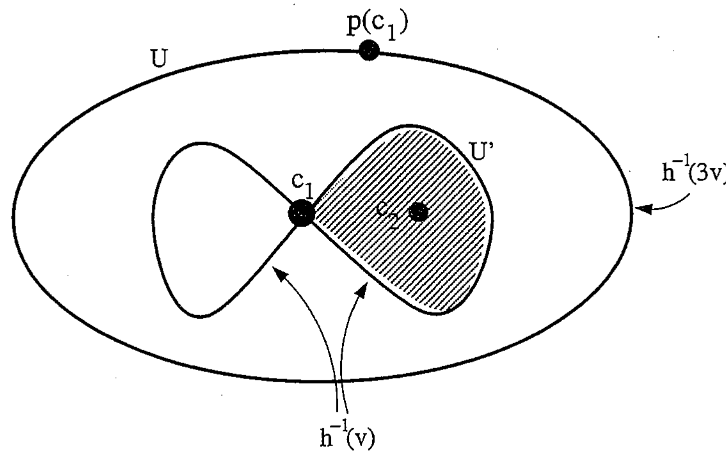



Figure 11. The levels sets of level $3v$ and $v$ for $h$ along with the domains $U'$ and $U$.

**Example 3.** Let $f(z) = -2 + \cos z$ and $U' = \{z \mid |\operatorname{Re} z| < 2,\ |\operatorname{Im} z| < 3\}$. The map $f|U'$ is polynomial-like of degree 2 on $U'$ even though $f(z)$ is a transcendental, entire function.

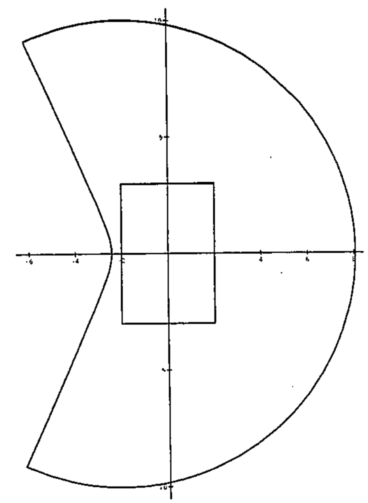


Figure 12. The rectangle $U'$ and its image $U$ under $f(z) = -2 + \cos z$. On $U'$, the transcendental function $f(z)$ is polynomial-like of degree 2.

The filled Julia set of a polynomial is the set of points whose orbits do not converge to infinity. A similar concept exists for polynomial-like maps.

**Definition.** If $f : U' \to U$ is a polynomial-like map, then the *filled Julia set* $K_f$ for $f$ is

$$K_f = \{z \in U \mid f^n(z) \in U' \text{ for all } n \in \mathbb{Z}^+\}.$$

**Theorem 3.** (Douady-Hubbard) *Associated to each polynomial-like map $f$ is a polynomial $q$ such that the dynamics of $f$ on a neighborhood of $K_f$ is topologically conjugate to the dynamics of $q$ on a neighborhood of $K_q$.* ∎

This theorem explains why the black region in Figure 9 resembles the black region in Figure 10. The map $N^2$ is polynomial-like of degree 2 on a region $U'$ contained in the rectangle shown in Figure 9. The corresponding quadratic given by Theorem 3 is $z \mapsto z^2 - 1$.

Although Theorem 3 explains why we see quadratic Julia sets in Newton's method, it does not explain why there is a region in Figure 7 that resembles the Mandelbrot set. However, the Douady-Hubbard paper also contains results that apply to parameter space.

The following example illustrates their result in a familiar case.

**Example 4.** Let $f_\lambda$ be the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$. Since any quadratic is conjugate to one of the form $z \mapsto z^2 + c$, we have $c$ as a function of $\lambda$. In fact, this function is

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4},$$

which is a branched covering map. ◇

With Example 4 in mind, we turn to the general result. Let $\Lambda$ be a simply-connected domain in $\mathbb{C}$ and $\{f_\lambda \mid \lambda \in \Lambda\}$ be a one-parameter family of degree-two, polynomial-like maps. We define $\Psi\colon \Lambda \to \mathbb{C}$ where the quadratic polynomial

$$q_\lambda(z) = z^2 + \Psi(\lambda)$$

is related to $f_\lambda$ by Theorem 3. We also define $M_\Lambda$ as $\Psi^{-1}(M)$.

**Theorem 4.** (Douady-Hubbard) *If $\Psi$ is not constant and $M_\Lambda$ is compact, then the map $\Psi\colon M_\Lambda \to M$ is a branched cover.* ∎

**Remark.** Theorem 4 explains why we see a Mandelbrot set in Figure 7. On a simply-connected domain contained in the rectangle illustrated in Figure 7, the family of second iterates of the Newton's method functions is a polynomial-like family of degree 2 (on the appropriate domains in $\overline{\mathbb{C}}$). Thus, Theorem 4 indicates that we will see branched covers of the Mandelbrot set in parameter space.

The Douady-Hubbard paper also indicates how to determine the degree of the covering map from Theorem 4. Let $A$ be a closed subset homeomorphic to $\overline{\mathbb{D}}$ such that $A \subset \Lambda$ and $M_\Lambda \subset \text{int}(A)$.



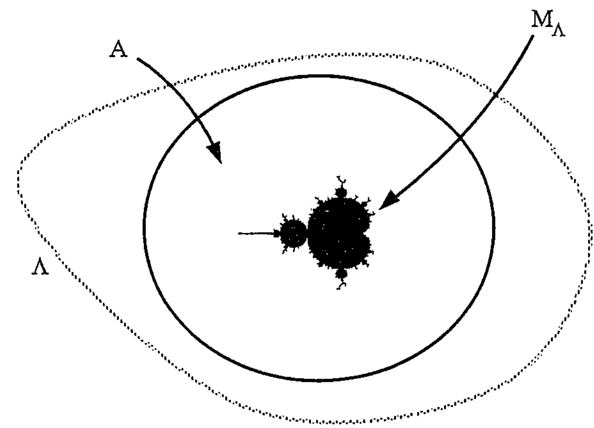



Figure 13. The closed, simply-connected region $A$ that contains $M_\Lambda$.

**Theorem 5.** *Let $f_\lambda$ and $\Psi$ represent the maps in Theorem 4 and let $w_\lambda$ denote the critical point of $f_\lambda$. Then, $\deg(\Psi\colon M_\Lambda \to M)$ equals the winding number of $f_\lambda(w_\lambda) - w_\lambda$ about 0 as $\lambda$ traverses once around the boundary of $A$.* ∎

Therefore, if the winding number is 1, $M_\Lambda$ is homeomorphic to the Mandelbrot set.

Polynomials whose degree is greater than three usually have move than one inflection point. Thus, examples of Newton's method with more than one periodic attracting orbit are possible. Hurley [H] has shown that, for $d \geq 3$, there is a polynomial of degree $d$ with $d - 2$ distinct attracting periodic orbits.

**Theorem 6.** (Hurley) *For each $d \geq 3$, there exists a polynomial $p(z)$ of degree $d$ whose corresponding Newton's method function has $d - 2$ distinct attracting periodic orbits of period greater than one.* ∎

## Appendix

The following list provides detailed specifications for Figures 5 – 9.

Figure 5:

| | | |
|---|---|---|
| | Lower left corner: | $-0.1$ |
| | Upper right corner: | $1.1 + 1.2i$ |

Figure 6:

| | | |
|---|---|---|
| | Lower left corner: | $0.869039 + 0.323172i$ |
| | Upper right corner: | $0.971966 + 0.450939i$ |

Figure 7:

| | | |
|---|---|---|
| | Lower left corner: | $0.894704 + 0.407193i$ |
| | Upper right corner: | $0.919835 + 0.435468i$ |

Figure 8:

| | | |
|---|---|---|
| | Lower left corner: | $-0.790167 - 0.907198i$ |
| | Upper right corner: | $1.83213 + 1.55605i$ |

Figure 9:

| | |
|---|---|
| Lower left corner: | $0.471402 - 0.023518i$ |
| Upper right corner: | $0.807454 + 0.318906i$ |

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# The Spider Algorithm

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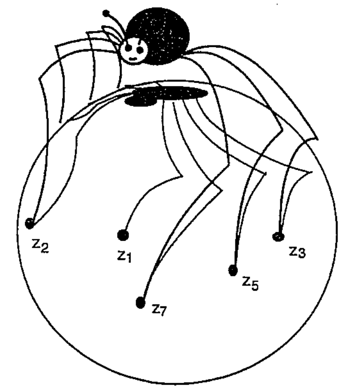


Charlotte [W] casts a 43/255-shadow.

One of the reasons complex analytic dynamics has been such a successful subject is the deep relation that has surfaced between conformal mapping, dynamics and combinatorics. The object of the spider algorithm is to construct polynomials with assigned combinatorics.

This shows up when you try to understand the Mandelbrot set. For this discussion we will write our quadratic polynomials $Q_c(z) = z^2 + c$. Every such polynomial has a *filled in Julia set* $K_c$, formed of the points with bounded orbits under iteration of $Q_c$.

A result of Fatou asserts that if the critical point $0 \in K_c$, then $K_c$ is connected, and if $0 \notin K_c$, then $K_c$ is a Cantor set. By definition, the Mandelbrot set $M$ is the set of $c$ for which $K_c$ is connected.

Let $\mathbb{D}$ denote the open unit disc, and let $\Phi_M : \overline{\mathbb{C}} - M \to \overline{\mathbb{C}} - \mathbb{D}$ be the conformal mapping which maps $\infty$ to $\infty$ and is tangent to the identity at infinity. The existence of this mapping is not obvious; it is proved to exist at the same time as the

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