Math 546: Complex dynamics
Homework 3
Tuesday, February 16, 2010

1. Show that the Julia set of $z^2 - 10$ is a Cantor set in the real line. Use any method you wish. The same is true of $z^2 + c$ for any real $c < -2$. (Compare Milnor Problem 4-e.) Sketch the level sets of $|\varphi|$ on the basin of infinity, where φ is the Böttcher uniformization near infinity.

2. (Milnor Problem 7-c) The Chebyshev polynomials are defined inductively by $P_1(z) = z$, $P_2(z) = z^2 - 2$, and

$$P_n(z) = zP_{n-1}(z) - P_{n-2}(z)$$

for all $n > 2$.

(a) Show that $P_n(w + w^{-1}) = w^n + w^{-n}$, or equivalently that $P_n(2\cos\theta) = 2\cos(n\theta)$.
(b) Show that $P_m \circ P_n = P_{mn}$.
(c) For each $n \geq 2$, show that the Julia set $J(P_n)$ is the interval $[-2, 2]$.
(d) For each $n \geq 3$, show that P_n has $n - 1$ distinct critical points but only two critical values, ± 2.

3. (Milnor Problem 7-e) Show that the Chebyshev polynomial of degree d and the power map z^d satisfy: for all but finitely many periodic points z_0, the multiplier $\lambda = (f^p)'(z_0)$ (where z_0 has period p) satisfies $|\lambda| = d^n$. In fact, these are the only polynomials with this property, up to conformal conjugacy!

4. (Milnor Problem 9-f) Consider a polynomial $f(z) = z^2 + c$ with c not in the Mandelbrot set, so $f^n(0) \to \infty$ as $n \to \infty$. Let φ be the Böttcher isomorphism near ∞, so that $\varphi f \varphi^{-1}(w) = w^2$ and $\varphi(\infty) = 0$. Let $U = \mathbb{C} \setminus \{z : |\varphi(z)| < |\varphi(c)|\}$. Show that U is conformally isomorphic to a disk and that $f^{-1}(U)$ has two connected components, each with closure contained in U. There is a well-defined inverse branch of f from U to each of these components, call them g_0 and g_1. Show that each g_j strictly contracts the hyperbolic metric on U. Use the g_j to show that the Julia set $J(f)$ is a Cantor set, canonically homeomorphic to the sequence space

$$\Sigma_2 = \prod_{n=0}^{\infty} \{0, 1\}$$

via a homeomorphism which conjugates $f|J(f)$ to the shift map

$$\sigma(x_0 x_1 x_2 \cdots) = x_1 x_2 x_3 \cdots .$$

In fact, the same is true for a polynomial of any degree d with all critical points in its basin of infinity; the polynomial $f|J(f)$ is topologically conjugate to the *one-sided full shift on d symbols*. The proof is similar to the degree 2 case.

5. Present your favorite solution to Hexi in his office hours: Mondays 11-12 or Wednesdays 12-1.