

DYNAMICS OF RATIONAL MAPS: A CURRENT ON THE BIFURCATION LOCUS

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ABSTRACT. Let $f_\lambda : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a family of rational maps of degree $d > 1$, parametrized holomorphically by λ in a complex manifold X . We show that there exists a canonical closed, positive (1,1)-current T on X supported exactly on the bifurcation locus $B(f) \subset X$. If X is a Stein manifold, then the stable regime $X - B(f)$ is also Stein. In particular, each stable component in the space Poly_d (or Rat_d) of all polynomials (or rational maps) of degree d is a domain of holomorphy.

1. INTRODUCTION

It is well-known that for a rational map $f : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ of degree $d > 1$, there is a natural f -invariant measure μ_f supported on the Julia set of f [B],[Ly]. This measure can be described as the weak limit of purely atomic measures,

$$\mu_f = \lim_{n \rightarrow \infty} \frac{1}{d^n} \sum_{\{z: f^n(z)=a\}} \delta_z,$$

for any $a \in \mathbf{P}^1$ (with at most two exceptions).

There is also a potential-theoretic description of μ_f , defined in terms of a homogeneous polynomial lift $F : \mathbf{C}^2 \rightarrow \mathbf{C}^2$ of f . The potential function on \mathbf{C}^2 is given by

$$(1) \quad h(z) = \lim_{m \rightarrow \infty} \frac{1}{d^m} \log \|F^m(z)\|,$$

and the (1,1)-current $\partial\bar{\partial}h$ satisfies

$$\pi^* \mu_f = \frac{i}{\pi} \partial\bar{\partial}h$$

where π is the canonical projection $\mathbf{C}^2 - \{0\} \rightarrow \mathbf{P}^1$ [HP]. In particular, when f is a monic polynomial, this definition reduces to

$$\mu_f = \frac{i}{\pi} \partial\bar{\partial}G = \frac{1}{2\pi} \Delta G \, dx \wedge dy,$$

where $G : \mathbf{C} \rightarrow [0, \infty)$ is the Green's function for the complement of the filled Julia set $K(f) = \{z : f^n(z) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$.

In this paper, we construct a (1,1)-current on the **parameter space** of a holomorphic family of rational maps, supported exactly on the bifurcation locus (just as μ_f is supported exactly on the Julia set).

Let X be a complex manifold. A **holomorphic family of rational maps f over X** is a holomorphic map $f : X \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$. For each parameter $\lambda \in X$, we obtain a rational map $f_\lambda : \mathbf{P}^1 \rightarrow \mathbf{P}^1$ with Julia set $J(f_\lambda)$. The **bifurcation locus $B(f)$** of the family f over X is the set of all $\lambda_0 \in X$ for which $\lambda \mapsto J(f_\lambda)$ is a discontinuous function (in the Hausdorff topology) in any neighborhood of λ_0 (§2).

Theorem 1.1. *Let $f : X \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a holomorphic family of rational maps on \mathbf{P}^1 of degree $d > 1$. Then there exists a canonical closed, positive (1,1)-current $T(f)$ on X such that the support of $T(f)$ is $B(f)$, the bifurcation locus of f .*

By general properties of positive currents (Lemma 3.3), we have

Corollary 1.2. *If X is a Stein manifold, then $X - B(f)$ is also Stein.*

Let Rat_d and Poly_d denote the “universal families” of all rational maps and of all monic polynomials of degree exactly $d > 1$. We have $\text{Poly}_d \simeq \mathbf{C}^d$ and $\text{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is a resultant hypersurface. In particular, Rat_d and Poly_d are Stein manifolds.

Corollary 1.3. *Every stable component in Rat_d and Poly_d is a domain of holomorphy (i.e. a Stein open subset).*

Corollary 1.3 answers a question posed by McMullen in [M2], motivated by analogies between rational maps and Teichmüller space. Bers and Ehrenpreis showed that finite-dimensional Teichmüller spaces are domains of holomorphy [BE].

Sketch proof of Theorem 1.1. Consider a holomorphic family of homogeneous polynomial maps $\{F_\lambda\}$ on \mathbf{C}^2 , locally lifting the holomorphic family f over X . Let $\{h_\lambda\}$ be the corresponding potential functions on \mathbf{C}^2 defined by equation (1). The function $h_\lambda(z)$ is plurisubharmonic in both $\lambda \in X$ and $z \in \mathbf{C}^2$, and it is pluriharmonic in z away from $\pi^{-1}(J(f_\lambda))$. Suppose for simplicity that we have holomorphic functions $c_j : X \rightarrow \mathbf{P}^1$, $j = 1, \dots, 2d-2$, parametrizing the critical points of f_λ in \mathbf{P}^1 . We choose lifts \tilde{c}_j from a neighborhood in X to \mathbf{C}^2 so that $c_j = \pi \circ \tilde{c}_j$ and define the plurisubharmonic function

$$H(\lambda) = \sum_j h_\lambda(\tilde{c}_j(\lambda)).$$

The desired (1,1)-current on X is defined by

$$T(f) = \frac{i}{\pi} \partial \bar{\partial} H,$$

independent of the choices of $\{F_\lambda\}$ and \tilde{c}_j . It is supported on $B(f)$ since H fails to be pluriharmonic exactly when a critical point $c_j(\lambda)$ passes through the Julia set $J(f_\lambda)$. \square

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2. STABILITY

Let $f : X \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a holomorphic family of rational maps of degree $d > 1$. The Julia sets of such a family are said to **move holomorphically** at a point $\lambda_0 \in X$ if there is a family of injections $\phi_\lambda : J_{\lambda_0} \rightarrow \mathbf{P}^1$, holomorphic in λ near λ_0 with $\phi_{\lambda_0} = \text{id}$, such that $\phi_\lambda(J_{\lambda_0}) = J_\lambda$ and $\phi_\lambda \circ f_{\lambda_0}(z) = f_\lambda \circ \phi_\lambda(z)$. In other words, ϕ_λ provides a conjugacy between f_{λ_0} and f_λ on their Julia sets. The family of rational maps f over X is **stable** at $\lambda_0 \in X$ if any of the following equivalent conditions are satisfied [M1, Theorem 4.2]:

- (1) The number of attracting cycles of f_λ is locally constant at λ_0 .
- (2) The maximum period of an attracting cycle of f_λ is locally bounded at λ_0 .
- (3) The Julia set moves holomorphically at λ_0 .
- (4) For all λ sufficiently close to λ_0 , every periodic point of f_λ is attracting, repelling, or persistently indifferent.
- (5) The Julia set J_λ depends continuously on λ (in the Hausdorff topology) in a neighborhood of λ_0 .

Suppose also that each of the $2d - 2$ critical points of f_λ are parametrized by holomorphic functions $c_j : X \rightarrow \mathbf{P}^1$. Then the following conditions are equivalent to those above:

- (6) For each j , the family of functions $\{\lambda \mapsto f_\lambda^n(c_j(\lambda))\}_{n \geq 0}$ is normal in some neighborhood of λ_0 .
- (7) For all nearby λ , $c_j(\lambda) \in J_\lambda$ if and only if $c_j(\lambda_0) \in J_{\lambda_0}$.

We let $S(f) \subset X$ denote the set of stable parameters and define the **bifurcation locus** $B(f)$ to be the complement $X - S(f)$. Mañé, Sad, and Sullivan showed that $S(f)$ is open and dense in X [MSS, Theorem A].

Example. In the family $f_c(z) = z^2 + c$, the bifurcation locus is $B(f) = \partial M$, where $M = \{c \in \mathbf{C} : f_c^n(0) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$ is the Mandelbrot set [M1, Theorem 4.6].

Lemma 2.1. *If $B(f)$ is contained in a complex hypersurface $D \subset X$, then $B(f)$ is empty.*

Proof. Suppose there exists $\lambda_0 \in B(f)$. By characterization (4) of stability, any neighborhood U of λ_0 must contain a point λ_1 at which the multiplier $m(\lambda)$ of a periodic cycle for f_λ is passing through the unit circle. In other words, the holomorphic function $m(\lambda)$ defined in a neighborhood N of λ_1 is non-constant with $|m(\lambda_1)| = 1$. The set $\{\lambda \in N : |m(\lambda)| = 1\}$ lies in the bifurcation locus and cannot be completely contained in a hypersurface. \square

3. STEIN MANIFOLDS AND POSITIVE CURRENTS

Let X be a paracompact complex manifold and $\mathcal{O}(X)$ its ring of holomorphic functions. Then X is a **Stein manifold** if the following three conditions are satisfied:

- for any $x \in X$ there exists a neighborhood U of x and $f_1, \dots, f_n \in \mathcal{O}(X)$ defining local coordinates on U ;
- for any $x \neq y \in X$, there exists an $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$;
and
- for any compact set K in X , the holomorphic hull

$$\hat{K} = \{x \in X : |f(x)| \leq \sup_K |f| \text{ for all } f \in \mathcal{O}(X)\}$$

is also compact in X .

An open domain Ω in X is **locally Stein** if every boundary point $p \in \partial\Omega$ has a neighborhood U such that $U \cap \Omega$ is Stein.

Properties of Stein manifolds. The Stein manifolds are exactly those which can be embedded as closed complex submanifolds of \mathbf{C}^N . If Ω is an open domain in \mathbf{C}^n then Ω is Stein if and only if Ω is pseudoconvex if and only if Ω is a domain of holomorphy. An open domain in a Stein manifold is Stein if and only if it is locally Stein. Also, an open domain in complex projective space \mathbf{P}^n is Stein if and only if it is locally Stein and not all of \mathbf{P}^n . See, for example, [H] and the survey article by Siu [S].

Examples. (1) \mathbf{C}^N is Stein. (2) The space of all monic polynomials of degree d , $\text{Poly}_d \simeq \mathbf{C}^d$, is Stein. (3) $\mathbf{P}^n - V$ for a hypersurface V is Stein. If V is the zero locus of degree d homogeneous polynomial F and $\{g_j\}$ a basis for the vector space of homogeneous polynomials of degree d , then the map $(g_1/F, \dots, g_N/F)$ embeds $\mathbf{P}^n - V$ as a closed complex submanifold of \mathbf{C}^N . (4) The space Rat_d of all rational maps $f(z) = P(z)/Q(z)$ on \mathbf{P}^1 of degree exactly d is Stein. Indeed, parameterizing f by the coefficients of P and Q defines an isomorphism $\text{Rat}_d \simeq \mathbf{P}^{2d+1} - V$, where V is the resultant hypersurface given by the condition $\gcd(P, Q) \neq 1$.

A (p, q) -**current** T on a complex manifold of dimension n is an element of the dual space to smooth $(n-p, n-q)$ -forms with compact support. See [HP], [Le], and [GH] for details. The wedge product of a (p, q) -current T with any smooth $(n-p, n-q)$ -form α defines a distribution by $(T \wedge \alpha)(f) = T(f\alpha)$ for $f \in C_c^\infty(X)$. Recall that a distribution δ is positive if $\delta(f) \geq 0$ for functions $f \geq 0$. A (p, p) -current is **positive** if for any system of $n-p$ smooth $(1,0)$ -forms with compact support, $\{\alpha_1, \dots, \alpha_{n-p}\}$, the product

$$T \wedge (i\alpha_1 \wedge \bar{\alpha}_1) \wedge \dots \wedge (i\alpha_{n-p} \wedge \bar{\alpha}_{n-p})$$

is a positive distribution.

An upper-semicontinuous function h on a complex manifold X is **plurisubharmonic** if $h|_{\mathbf{D}}$ is subharmonic for any complex analytic disk \mathbf{D}^1 in X . The current $T = i\partial\bar{\partial}h$ is positive for any plurisubharmonic h , and $T \equiv 0$

if and only if h is pluriharmonic. The “ $\partial\bar{\partial}$ -Poincaré Lemma” says that any closed, positive (1,1)-current T on a complex manifold is locally of the form $i\partial\bar{\partial}h$ for some plurisubharmonic function h [GH].

The next three Lemmas show that the “region of pluriharmonicity” of a plurisubharmonic function is locally Stein. See [C, Theorem 6.2], [U, Lemma 2.4], [FS, Lemma 5.3], and [R, Theorem II.2.3] for similar statements.

Lemma 3.1. *Suppose h is plurisubharmonic on the open unit polydisk \mathbf{D}^2 in \mathbf{C}^2 and h is pluriharmonic on the “Hartogs domain”*

$$\Omega_\delta = \{(z, w) : |z| < 1, |w| < \delta\} \cup \{(z, w) : 1 - \delta < |z| < 1, |w| < 1\}.$$

Then h is pluriharmonic on \mathbf{D}^2 .

Proof. Let H be a holomorphic function on Ω_δ such that $h = \operatorname{Re} H$. Any holomorphic function on Ω_δ extends to \mathbf{D}^2 , and extending H we have $h \leq \operatorname{Re} H$ on \mathbf{D}^2 since h is plurisubharmonic. The set $A = \{z \in \mathbf{D}^2 : h = \operatorname{Re} H\}$ is closed by upper-semi-continuity of h . If A has a boundary point $w \in \mathbf{D}^2$, then for any ball $B(w)$ about w , we have

$$\begin{aligned} h(w) &= \operatorname{Re} H(w) \\ &= \frac{1}{|B(w)|} \int_{B(w)} \operatorname{Re} H \\ &> \frac{1}{|B(w)|} \int_{B(w)} h \end{aligned}$$

since $\operatorname{Re} H > h$ on a set of positive measure in $B(w)$. This inequality, however, contradicts the sub-mean-value property of the subharmonic function h . Therefore $A = \mathbf{D}^2$ and h is pluriharmonic on the polydisk. \square

Lemma 3.2. *Let X be a complex manifold. If an open subset $\Omega \subset X$ is not locally Stein, there is a $\delta > 0$ and an embedding*

$$e : \mathbf{D}^2 \rightarrow X$$

so that $e(\Omega_\delta) \subset \Omega$ but $e(\mathbf{D}^2) \not\subset \Omega$.

Proof. Suppose Ω is not locally Stein at $x \in \partial\Omega$. By choosing local coordinates in a Stein neighborhood U of x in X , we may assume that U is a pseudoconvex domain in \mathbf{C}^n . Then $\Omega_0 = U \cap \Omega$ is pseudoconvex and the function $\phi(z) = -\log d_0(z)$ is not plurisubharmonic near $x \in \partial\Omega$. Here, d_0 is the Euclidean distance function to the boundary of Ω_0 .

If ϕ is not plurisubharmonic at the point $z_0 \in U \cap \Omega$, then there is a one-dimensional disk $\alpha : \mathbf{D}^1 \rightarrow \Omega$ centered at z_0 such that $\int_{\partial\mathbf{D}^1} \phi < \phi(z_0)$ (identifying the disk with its image $\alpha(\mathbf{D}^1)$). Let ψ be a harmonic function on \mathbf{D}^1 so that $\psi = \phi$ on $\partial\mathbf{D}^1$. Then $\psi(z_0) < \phi(z_0)$. Let Ψ be a holomorphic function on \mathbf{D}^1 with $\psi = \operatorname{Re} \Psi$.

Now, let $p \in \partial\Omega$ be such that $d_0(z_0) = |z_0 - p|$. Let $e : \mathbf{D}^2 \rightarrow U$ be given by

$$e(z_1, z_2) = \alpha(z_1) + z_2(1 - \varepsilon)e^{-\Psi(z_1)}(p - z_0).$$

That is, the two-dimensional polydisk is embedded so that at each point $z_1 \in \mathbf{D}^1$ there is a disk of radius $|(1 - \varepsilon) \exp(-\Psi(z_1))|$ in the direction of $p - z_0$. If ε is small enough we have a Hartogs-type subset of the polydisk contained in Ω but the polydisk is not contained in Ω since $d_0(z_0, \partial\Omega) = \exp(-\phi(z_0)) < \exp(-\psi(z_0))$. \square

Lemma 3.3. *Let T be a closed, positive $(1,1)$ -current on a complex manifold X . Then $\Omega = X - \text{supp}(T)$ is locally Stein.*

Proof. Let p be a boundary point of Ω . Choose a Stein neighborhood U of p in X so that $T = i\partial\bar{\partial}h$ for some plurisubharmonic function h on U . By definition of Ω , h is pluriharmonic on $U \cap \Omega$.

If Ω is not locally Stein at p , then by Lemma 3.2, we can embed a two-dimensional polydisk into U so that a Hartogs-type domain Ω_δ lies in Ω , but the polydisk is not contained in Ω . By Lemma 3.1, h must be pluriharmonic on the whole polydisk, contradicting the definition of Ω . \square

Corollary 3.4. *If X is Stein, then so is $X - \text{supp } T$.*

Example. If X is a Stein manifold and V a hypersurface, then $V = \text{supp } T$ for a positive $(1,1)$ -current T given locally by $T = \frac{i}{\pi} \partial\bar{\partial} \log |f|$, where V is the zero set of f . Lemma 3.3 shows that $X - V$ is locally Stein, and thus Stein. Similarly, $\mathbf{P}^n - V$ is Stein for any hypersurface V .

4. THE POTENTIAL FUNCTION OF A RATIONAL MAP

Let $f : \mathbf{P}^n \rightarrow \mathbf{P}^n$ be a holomorphic map. Let $F : \mathbf{C}^{n+1} \rightarrow \mathbf{C}^{n+1}$ be a lift of f to a homogeneous polynomial, unique up to scalar multiple, so that $\pi \circ F = f \circ \pi$ where π is the projection $\mathbf{C}^{n+1} \setminus \{0\} \rightarrow \mathbf{P}^n$. Let d be the degree of the components of F ; then f has topological degree d^n .

Assume that $d > 1$. Following [HP], we define the **potential function** of F by

$$h_F(z) = \lim_{m \rightarrow \infty} \frac{1}{d^m} \log \|F^m(z)\|.$$

The limit converges uniformly on compact subsets of $\mathbf{C}^{n+1} - 0$, and $h_F(z)$ is plurisubharmonic on \mathbf{C}^{n+1} since $\log \|\cdot\|$ is plurisubharmonic. Let $\Omega_F \subset \mathbf{C}^{n+1}$ be the basin of attraction of the origin for F ; that is,

$$\Omega_F = \{x \in \mathbf{C}^{n+1} : F^m(x) \rightarrow 0 \text{ as } m \rightarrow \infty\}.$$

Note that Ω_F is open and bounded.

From the definition, we obtain the following properties of the potential function h_F [HP]:

- (1) $h_F(\alpha z) = h_F(z) + \log |\alpha|$ for $\alpha \in \mathbf{C}^*$;
- (2) $\Omega_F = \{z : h_F(z) < 0\}$; and
- (3) h_F is independent of the choice of norm $\|\cdot\|$ on \mathbf{C}^{n+1} .

Theorem 4.1. (Hubbard-Papadopol, Ueda, Fornaess-Sibony) *The support of the positive (1,1)-current*

$$\omega_f = \frac{i}{\pi} \partial \bar{\partial} h_F$$

on $\mathbf{C}^{n+1} - 0$ is equal to the preimage of the Julia set $\pi^{-1}(J(f))$. If $n = 1$, then the Brodin-Lyubich measure μ_f satisfies $\pi^* \mu_f = \omega_f$.

Proof. See [HP, Theorem 4.1] for $n = 1$ and [U, Theorem 2.2], [FS, Theorem 2.12] for $n > 1$. □

From Corollary 3.4, we obtain the following ([U, Theorem 2.3], [FS, Theorem 5.2]):

Corollary 4.2. (Ueda, Fornaess-Sibony) *The Fatou components of $f : \mathbf{P}^n \rightarrow \mathbf{P}^n$ are Stein.*

5. THE BIFURCATION CURRENT

In this section we complete the proof of Theorem 1.1. Let $f : X \times \mathbf{P}^1 \rightarrow \mathbf{P}^1$ be a holomorphic family of rational maps on \mathbf{P}^1 of degree $d > 1$. Let $\{F_\lambda\}$ be a holomorphic family of homogeneous polynomials on \mathbf{C}^2 , locally lifting the family f , and let h_λ denote the potential function of F_λ (§4). The potential function $h_\lambda(z)$ is plurisubharmonic as a function of the pair (λ, z) .

Fix $\lambda_0 \in X$. In a neighborhood U of λ_0 , we can choose coordinates on \mathbf{P}^1 so that ∞ is not a critical point of f_λ , $\lambda \in U$. For $z \in \mathbf{P}^1 - \{\infty\}$, let $\tilde{z} = (z, 1) \in \mathbf{C}^2$. Define a function H on U by

$$H(\lambda) = \sum_{\{c: f'_\lambda(c)=0\}} h_\lambda(\tilde{c}),$$

where the critical points are counted with multiplicity. Now, let $N(\lambda)$ be the number of critical points of the rational map f_λ (counted without multiplicity). Let

$$D(f) = \{\lambda_0 \in X : N(\lambda) \text{ does not have a local maximum at } \lambda = \lambda_0\}.$$

Then $D(f)$ is a complex hypersurface in X , since it is defined by the vanishing of a discriminant. If $\lambda_0 \notin D(f)$, there exists a neighborhood U of λ_0 and holomorphic functions $c_j : U \rightarrow \mathbf{P}^1$, $j = 1, \dots, 2d - 2$, parametrizing the critical points of f_λ , such that $\infty \notin c_j(U)$ for all j . In this case, we can express H as the sum

$$H(\lambda) = \sum_j H_j(\lambda)$$

of the plurisubharmonic functions

$$\begin{aligned} H_j(\lambda) &= h_\lambda \circ \tilde{c}_j(\lambda) \\ &= \lim_{m \rightarrow \infty} \frac{1}{d^m} \log \|F_\lambda^m(\tilde{c}_j(\lambda))\|. \end{aligned}$$

For any $\lambda_0 \in X$, then, H is defined and continuous in a neighborhood U of λ_0 and plurisubharmonic on $U - D(f)$; therefore H is plurisubharmonic on U .

The **bifurcation current** T is the positive (1,1)-current on parameter space X given locally by

$$T = \frac{i}{\pi} \partial \bar{\partial} H.$$

The next Lemma shows that T is globally well-defined on X .

Lemma 5.1. *The current $T = \frac{i}{\pi} \partial \bar{\partial} H$ is independent of (a) the choice of lifts \tilde{c}_j of c_j and (b) the choice of lifts F_λ of f_λ .*

Proof. Suppose we define a new lift $\hat{c}_j(\lambda) = t(\lambda) \cdot \tilde{c}_j(\lambda)$ for some holomorphic function t taking values in \mathbf{C}^* . Property (1) of the potential function h_λ (§4) implies that $h_\lambda(\hat{c}_j(\lambda)) = h_\lambda(\tilde{c}_j(\lambda)) + \log |t(\lambda)|$ and $i\partial\bar{\partial}H$ is unchanged since $\log |t(\lambda)|$ is pluriharmonic, proving (a). If the lifted family $\{F_\lambda\}$ is similarly replaced by $\{t(\lambda) \cdot F_\lambda\}$, a computation shows that h_λ is changed only by the addition of the pluriharmonic term $\frac{1}{d-1} \log |t(\lambda)|$ where d is the degree of the f_λ . This proves (b). \square

Lemma 5.2. *A parameter λ_0 lies in the stable regime $S(f) \subset X$ if and only if the function H is pluriharmonic in a neighborhood of λ_0 .*

Proof. Let us first suppose that $\lambda_0 \in S(f)$ is not in $D(f)$ (in the notation above). By characterization (6) of stability (§2), for each j , the family of functions $\{\lambda \mapsto f_\lambda^m(c_j(\lambda))\}$ is normal in a neighborhood V of λ_0 ; hence, there exists a subsequence converging uniformly on compact subsets to a holomorphic function $g_j(\lambda)$. As in [HP, Prop 5.4], we can shrink our neighborhood V if necessary to find a norm $\|\cdot\|$ on \mathbf{C}^2 so that $\log \|\cdot\|$ is pluriharmonic on $\pi^{-1}(g_j(V))$; e.g., if $g_j(V)$ is disjoint from $\{|x| = |y|\}$, we can choose norm $\|(x, y)\| = \max\{|x|, |y|\}$. Then, on any compact set in V , the functions

$$\lambda \mapsto \frac{1}{d^{mk}} \log \|F_\lambda^{mk}(\tilde{c}_j(\lambda))\|$$

are pluriharmonic if k is large enough. By property (3) of the potential function h_λ (§4), this subsequence converges uniformly to H_j . Therefore, H is pluriharmonic on V .

If λ_0 lies in $D(f) \cap S(f)$, then H is defined and continuous on a neighborhood V of λ_0 and pluriharmonic on $V - D(f)$. As $D(f)$ has codimension 1, H must be pluriharmonic on all of V .

For the converse, let us suppose again that $\lambda_0 \notin D(f)$ and that H is pluriharmonic in a neighborhood of λ_0 . Each H_j is pluriharmonic and so we may write $H_j = \text{Re } G_j$ in a neighborhood V of λ_0 . In analogy with [U,

Prop. 2.1], we define new lifts $\hat{c}_j(\lambda) = e^{-G_j(\lambda)} \cdot \tilde{c}(\lambda)$ of the c_j and compute

$$\begin{aligned} h_\lambda(\hat{c}_j(\lambda)) &= h_\lambda(\tilde{c}(\lambda)) + \log |e^{-G_j(\lambda)}| \\ &= h_\lambda(\tilde{c}(\lambda)) - \operatorname{Re} G_j \\ &= H_j - H_j \\ &= 0. \end{aligned}$$

By property (2) of h_λ , this implies that $\hat{c}_j(\lambda)$ lies in $\partial\Omega_\lambda$ for all $\lambda \in V$. If V is small enough, the set $\cup_{\lambda \in V} (\{\lambda\} \times \partial\Omega_\lambda)$ has compact closure in $X \times \mathbf{C}^2$. As the functions F_λ preserve $\partial\Omega_\lambda$, the family $\{\lambda \mapsto F_\lambda^n(\hat{c}_j(\lambda))\}$ is uniformly bounded and thus normal. Of course, $f_\lambda^n \circ c_j = \pi \circ F_\lambda^n \circ \hat{c}_j$ demonstrating that λ_0 is a stable parameter by (6) of Section 2.

Finally suppose that H is pluriharmonic in a neighborhood U of parameter $\lambda_0 \in D(f)$. Then $U - D(f)$ lies in the stable regime and Lemma 2.1 shows that all of U must belong to $S(f)$. \square

Proof of Theorem 1.1. Let T be the bifurcation current defined above for the family of rational maps f over X . By Lemma 5.2, the support of T is the bifurcation locus $B(f)$. \square

Corollaries 1.2 and 1.3 now follow immediately from Corollary 3.4.

6. EXAMPLES

Example 6.1. In the family $\{f_\lambda(z) = z^d + \lambda\}$, $\lambda \in \mathbf{C}$, the bifurcation current T takes the form

$$T = \frac{d-1}{d} \left(\frac{i}{\pi} \partial \bar{\partial} G \right)$$

where G is the Green's function for the complement of the “degree d Mandelbrot set” $M_d = \{\lambda : f_\lambda^n(0) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$. That is, T is a multiple of harmonic measure supported on ∂M_d . The T -mass of ∂M_d is $(d-1)/d$.

Proof. If G_λ denotes the Green's function for the complement of the filled Julia set $K(f_\lambda) = \{z : f_\lambda^n(z) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}$, then $G(\lambda) = G_\lambda(\lambda)$ (see e.g. [CG, VIII.4]). By [HP, Prop 8.1], we have

$$h_\lambda(x, y) = G_\lambda(x/y) + \log |y|$$

where (x, y) , $y \neq 0$, is a point of \mathbf{C}^2 . Note that $d - 1$ of the critical points of f_λ are at $z = 0$ and the other $d - 1$ are at $z = \infty$. Computing, we find

$$\begin{aligned} T &= \frac{i}{\pi} \sum_j \partial \bar{\partial} h_\lambda(\tilde{c}_j(\lambda)) \\ &= (d - 1) \frac{i}{\pi} \partial \bar{\partial} h_\lambda(0, 1) \\ &= (d - 1) \frac{i}{\pi} \partial \bar{\partial} G_\lambda(0) \\ &= \frac{d - 1}{d} \frac{i}{\pi} \partial \bar{\partial} G_\lambda(\lambda). \end{aligned}$$

□

Example 6.2. Let f be a polynomial of degree d , μ_f the Brolin-Lyubich measure on the Julia set, and G_f the Green's function for the complement of the filled Julia set (defined in the previous example). The Lyapunov exponent of f satisfies ([Prz],[Mn])

$$L(f) = \log d + \sum_{\{c \in \mathbf{C}: f'(c)=0\}} G_f(c).$$

If $\{f_\lambda\}$ is any holomorphic family of polynomials, the Lyapunov exponent as a function of the parameter is a potential function for the bifurcation current; that is,

$$T = \frac{i}{\pi} \partial \bar{\partial} L.$$

In the sequel, we examine further the connection between the bifurcation current and the Lyapunov exponent.

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