# RIEMANNIAN SYMMETRIC SPACES AND BOUNDED DOMAINS IN $\mathbf{C}^n$

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## 0. Introduction

This paper is to serve as an introduction to the study of symmetric spaces, with the goal of describing Hermitian symmetric spaces of noncompact type. There are three basic types of symmetric space: compact, noncompact, and Euclidean, defined in terms of properties of $\mathfrak{g}$, the Lie algebra of its group of isometries. It turns out that a simply connected symmetric space can be described completely in terms of the pair $(\mathfrak{g}, s)$, where $s$ is an involutive automorphism of $\mathfrak{g}$. All symmetric spaces of noncompact and Euclidean type are simply connected, but the compact case is quite different and a full understanding is beyond the scope of this paper. Our main goal is to discuss the Hermitian symmetric spaces which are all simply connected, and in particular to prove that those of noncompact type are exactly the bounded symmetric domains in $\mathbf{C}^n$ with Riemannian structure given by the Bergman metric.

Section 1 gives basic definitions and properties of symmetric spaces with some examples at the end. Section 2 contains a summary of the basics of Lie algebras, and Section 3 provides an overview of the decomposition of symmetric spaces into compact, noncompact, and Euclidean type. We achieve our main goal in Section 4. It should be mentioned that a complete classification of symmetric spaces is possible using the classification of semisimple Lie algebras. While the details of the classification will not be carried out here, Section 5 outlines the basic strategy.

## 1. Symmetric Spaces

A Riemannian manifold $(M, g)$ is a **symmetric space** if for each $p \in M$ there exists an isometry $S$ fixing $p$ and such that $dS_p = -I$ on $T_pM$. Such an isometry is called a **symmetry about p**. As isometries preserve geodesics, we see that any geodesic $\gamma(t)$ with $\gamma(0) = p$ must satisfy $S(\gamma(t)) = \gamma(-t)$. It follows that $M$ is homogeneous. For, if two points in $M$ are joined by a geodesic $\gamma : [0,1] \to M$ then the symmetry about $\gamma(1/2)$ interchanges the endpoints. Any two points in $M$ can be joined by a broken geodesic, so composing such symmetries demonstrates homogeneity. By similar reasoning, we see that $M$ is complete. For if $\gamma : [0,1] \to M$ is a geodesic, we simply use the symmetry about $\gamma(1)$ to extend $\gamma$ to the interval $[0,2]$.

We note that if $M$ is connected, the symmetry about $p$ is uniquely defined. If $v \in T_pM$, $\gamma_v(t)$ will always denote the geodesic given by $\exp_p(tv)$.

**Proposition 1.1.** *Suppose $\phi$ and $\psi : M \to N$ are local isometries of Riemannian manifolds with $M$ connected and there exists $p \in M$ such that $\phi(p) = \psi(p)$ and $d\phi_p = d\psi_p$. Then $\phi = \psi$.*

*Proof.* Put $A = \{q \in M : \phi(q) = \psi(q) \text{ and } d\phi_q = d\psi_q\}$. By continuity, $A$ is closed in $M$. $A$ is nonempty so we must show that $A$ is open. Let $q \in A$ and $U$ a normal neighborhood of $q$. For $q' \in U$, there exists a $v \in T_qM$ such that $\gamma_v(1) = \exp_q(v) = q'$. Then $\phi(q') = \phi(\gamma_v(1)) = \gamma_{d\phi(v)}(1) = \gamma_{d\psi(v)}(1) = \psi(\gamma_v(1)) = \psi(q')$. Thus $\phi = \psi$ on $U$ and so $d\phi_{q'} = d\psi_{q'}$ for all $q' \in U$. [O, p.91] □

Below are a few examples of symmetric spaces.

- Lie group $G$ with bi-invariant metric. The symmetry about the identity is given by $S_e(g) = g^{-1}$. The symmetry for any other point $h \in G$ is given by $S_h(g) = h \cdot S_e(h^{-1}g) = hg^{-1}h$.
- Sphere $S^n$. For each $p \in S^n \subset \mathbf{R}^{n+1}$, choose an orthonormal basis $\{p, v_1, \ldots, v_n\}$ for $\mathbf{R}^{n+1}$ and with respect to this basis, put
$$S_p = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix} \in \mathcal{O}(n+1).$$
- Poincare model for the hyperbolic plane. With respect to the Poincare metric
$$\frac{2|dz|}{1-|z|^2}$$
the conformal automorphisms of the disk
$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$$
for $|\alpha| < 1$ are isometries. Let $S_0(z) = -z$ and put $S_\alpha = \phi_\alpha \circ S_0 \circ \phi_\alpha$.

## Locally symmetric spaces and the curvature tensor

On a Riemannian manifold $M$ let $\nabla$ be the Levi-Civita connection. The Riemann curvature tensor is defined by
$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

**Proposition 1.2.** *Any symmetric space has parallel curvature tensor.*

*Proof.* The isometries $S_p$ leave the curvature tensor and its covariant derivative invariant. We have $dS_p = -I$ on $T_pM$ so for $X, Y, Z \in T_pM$,
$$-(\nabla_v R)(X,Y)Z = \nabla_{-v} R(-X,-Y)(-Z) = (\nabla_v R)(X,Y)Z$$
and therefore $\nabla R \equiv 0$. [P, p.210] □

Let $M$ be a Riemannian manifold. For each $p \in M$, let $U$ be a normal neighborhood of $p$ and define $\zeta_p(x) = \exp_p(-\exp_p^{-1}(x))$ on $U$. This $\zeta_p$ is known as the **local geodesic symmetry** at $p$ since it clearly takes geodesic $\gamma_v(t)$ to $\gamma_v(-t)$ for small $t$:

$$\zeta_p(\gamma_v(t)) = \zeta_p(\exp(tv)) = \exp(-tv) = \gamma_v(-t).$$

Note that $d\zeta_p = -I$. [O, p.223]. M is said to be **locally symmetric** if the local geodesic symmetry at each point is an isometry.

**Theorem 1.3.** *$M$ is locally symmetric if and only if $\nabla R = 0$.*

*Proof.* One direction follows from Proposition 1.2 since any symmetric space is locally symmetric. We need only show that parallel curvature tensor implies that the local geodesic symmetries are isometries. Let $\zeta$ denoted the local geodesic symmetry at a point $p$. We hope to show that if $U$ is a normal neighborhood of $p \in M$ and $q \in U$, $v \in T_qM$, we want

$$\langle d\zeta_q v, d\zeta_q v\rangle = \langle v, v\rangle.$$

Let $V = \exp_p^{-1}(U)$ in $T_pM$. There exists a unique $x \in V$ so that $\gamma_x(1) = \exp_p(x) = q$ and a unique $w_x \in T_x(T_pM)$ so that $(d\exp_p)_x(w_x) = v$. Identify $w_x$ with a vector $w \in T_pM$.

Let $J$ be the Jacobi vector field along $\gamma_x$ with $J(0) = 0$ and $J'(0) = w \in T_pM$. Then $J(1) = v$. Indeed, we consider the map $(s,t) \mapsto t(x+sw)$ in $T_pM$ for $t \in [0,1]$ and $|s|$ small. Then $(s,t) \mapsto \exp_p(t(x+sw))$ defines a variation of geodesics of the geodesic $\gamma_x(t)$. The vector field $W = \frac{d}{ds}\exp_p(t(x+sw))|_{s=0}$ is Jacobi along $\gamma_x$. Note that W equals $v = (d\exp_p)_x(w_x)$ at time $t = 1$. We aim to show that this Jacobi field $W$ is our Jacobi field $J$. The curve $s \mapsto t(x+sw)$ is constant at $t = 0$ so $W(0) = 0$. Note that $W'(0) = \frac{d}{dt}\frac{d}{ds}\exp_p(t(x+sw))|_{s,t=0} = \frac{d}{ds}\frac{d}{dt}\exp_p(t(x+sw))|_{t,s=0}$. Of course, $s \mapsto \frac{d}{dt}\exp_p(t(x+sw))|_{t=0}$ is a vector field along the constant curve at $p$. Therefore $W'(0) = w$ and so $W = J$.

Now we have that $d\zeta_p = -I$ and putting $w_{-x} = d(d\zeta_p)_x(w_x)$ in $T_{-x}(T_pM)$, we may identify $w_{-x}$ with $-w$ in $T_pM$. By the definition of $\zeta$ we have that $d\zeta_q(v) = (d\exp_p)_{-x}(w_{-x})$. Let $J_-$ be the Jacobi field along $\gamma_{-x}$ with $J_-(0) = 0$ and $J_-'(0) = -w$. We have so far $\langle J(1), J(1)\rangle = \langle v, v\rangle$ and $\langle J_-(1), J_-(1)\rangle = \langle d\zeta_q v, d\zeta_q v\rangle$.

Let $E_1, \ldots, E_n$ be an orthonormal basis for $T_pM$ and extend to parallel vector fields along $\gamma_x(t)$. Also, parallel translate $-E_1, \ldots, -E_n$ along $\gamma_{-x}(t)$. Write $\gamma_x'(t) = \sum c_i E_i(t)$ along $\gamma_x$ with constant coefficients and $J(t) = \sum a_i(t)E_i(t)$. By the Jacobi equation, the $a_i$ satisfy

$$\frac{d^2}{dt^2}a_m(t) = \sum_{ijk} R^m_{ijk}(t)c_i a_j(t) c_k$$

where $R^m_{ijk} = \langle R(E_j, E_k)E_i, E_m\rangle$. Similarly write $J_-(t)$ as $\sum b_i(t)(-E_i(t))$ along $\gamma_{-x}$ and the differential equation for the coefficients $b_i$ to obtain terms

$\hat{R}^m_{ijk} = \langle R(-E_j, -E_k) - E_i, -E_m\rangle$. Note that $\nabla R = 0$ implies these terms $R^m_{ijk}(t)$ and $\hat{R}^m_{ijk}(t)$ remain constant, and since $R^m_{ijk}(0) = \hat{R}^m_{ijk}(0)$ we must have $R^m_{ijk} \equiv \hat{R}^m_{ijk}$. As a consequence, the $a_i$ and $b_i$ satisfy the same differential equation with the same initial conditions and so must be equal. Therefore,

$$\langle d\zeta_q v, d\zeta_q v\rangle = \langle J_-(1), J_-(1)\rangle = \langle J(1), J(1)\rangle = \langle v, v\rangle.$$

[O, p.222] □

If the given manifold $M$ is simply connected and complete, then parallel curvature tensor is enough to show that $M$ is symmetric.

**Theorem 1.4.** *A complete, simply connected, locally symmetric space is symmetric.*

See [O, p.224] for the proof of this theorem. He in fact shows the following: Let $M$ and $M'$ be complete, connected, locally symmetric Riemannian manifolds with $M$ simply connected. If $L : T_pM \to T_pM$ is a linear isometry that preserves curvature, then there is a unique Riemannian covering map $\phi : M \to M'$ such that $d\phi_p = L$. It follows that the local symmetries $\zeta_p$ are actually globally defined if $M$ is simply connected and complete.

### The symmetric pair

Let $M$ be a symmetric space. Since $M$ is homogeneous, the isometry group $I(M)$ acts transitively and so does the identity component $G = I_0(M)$. (The subscript 0 will always denote the component of the identity.) We may write $M = G/H$, where $H$ is the isotropy group for any point $p \in M$. Note that $H$ is a compact subgroup of $G$ (see for example Helgason). If $S$ is the symmetry about $p$, define $\sigma$ on $G$ by $\sigma(g) = SgS$. This map $\sigma$ is an involutive automorphism of $G$ since $S$ is involutive and $\sigma$ is merely conjugation by $S$. The set $F^\sigma = \{g \in G : \sigma(g) = g\}$ is a closed subgroup of $G$. Let us note that the isotropy group $H$ satisfies $F^\sigma_0 \subset H \subset F^\sigma$. Indeed, if $h \in H$, the differential of $\sigma(h)$ at $p$ is given by $dS_p dh_p dS_p = dh_p$ since $dS_p = -I$. Since $M$ is connected, Proposition 1.1 shows that $\sigma(h) = h$. Thus $H \subset F^\sigma$. For $F^\sigma_0 \subset H$ note first that $F^\sigma_0$ is generated by the one parameter subgroups $\alpha(t)$ of $F^\sigma_0$. It suffices to show that $\alpha(t)$ lies in $H$. We have $\sigma(\alpha(t)) = \alpha(t)$ so $S\alpha(t) = \alpha(t)S$ and thus $S(\alpha(t)p) = \alpha(t)S(p) = \alpha(t)p$. But $p$ is an isolated fixed point for $S$ and so we must have $\alpha(t)p = p$ for small $t$ and hence all $t$. Therefore $\alpha(t)$ lies in $H$. [O, p.315]

From this we formulate a definition. The pair $(G, H)$ is called a **symmetric pair** if there exists an involutive automorphism $\sigma$ of $G$ so that $F^\sigma_0 \subset H \subset F^\sigma$ and if the group $\mathrm{Ad}_G(H)$ is compact.

**Theorem 1.5.** *Let $(G, H)$ be a symmetric pair and $\pi$ the natural map $G \to G/H$. Let $\sigma$ be any involutive automorphism of $G$ such that $F^\sigma_0 \subset H \subset F^\sigma$. Then any $G$-invariant Riemannian metric on $G/H$ makes $G/H$ into a* *symmetric space with the symmetry $S$ about $p = \pi(e)$ satisfying $S \circ \pi = \pi \circ \sigma$. If $\tau_g : xH \mapsto gxH$ then $\tau_{\sigma(g)} = S\tau_g S$ for all $g \in G$. We note that $S$ is independent of the choice of metric.*

We hold off on the proof of this theorem in order to first point out the relationship between the involutive automorphism $\sigma$ and the Lie algebra $\mathfrak{g}$ of $G$.

**Proposition 1.6.** *Let $(G, H)$ be a symmetric pair and $\sigma$ an involutive automorphism of $G$ such that $F_0^\sigma \subset H \subset F^\sigma$. Then*

(1) $\mathfrak{h} = \{X \in \mathfrak{g} : d\sigma(X) = X\}$ *is the Lie algebra of $H$.*
(2) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ *a direct sum of vector spaces, where $\mathfrak{p} = \{X \in \mathfrak{g} : d\sigma(X) = -X\}$.*
(3) $\mathrm{Ad}_h(\mathfrak{p}) \subset \mathfrak{p}$ *for all $h \in H$.*
(4) $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$.

*Proof.* For (1) $\sigma|H$ is the identity mapping so obviously, $d\sigma(X) = X$ for all $X$ in the Lie algebra of $H$. On the other hand, suppose $d\sigma(X) = X$. Let $\alpha$ be the one parameter subgroup of $H$ generated by $X$. Since $\sigma \circ \alpha$ is also a one parameter subgroup and $(\sigma \circ \alpha)'(0) = \alpha'(0)$ we must have $\sigma \circ \alpha = \alpha$. Thus $\alpha \subset F^\sigma$ and indeed $\alpha \subset F_0^\sigma \subset H$. Therefore $\mathfrak{h}$ is the Lie algebra of $H$.

For $X \in \mathfrak{g}$, put $X_\mathfrak{h} = 1/2(X + d\sigma X) \in \mathfrak{h}$ and $X_\mathfrak{p} = 1/2(X - d\sigma X) \in \mathfrak{p}$. Clearly $\mathfrak{h} \cap \mathfrak{p} = \{0\}$ so we have (2).

To see (3) we must show that $d\sigma(\mathrm{Ad}_h X) = -\mathrm{Ad}_h X$ for any $X \in \mathfrak{p}$ and $h \in H$. If $C_h$ denotes conjugation by $h$ on $G$, then $\sigma \circ C_h(a) = \sigma(hah^{-1}) = h\sigma(a)h^{-1} = C_h \circ \sigma(a)$ for any $a$ in $G$ since $\sigma(h) = h$. Thus,

$$d\sigma(\mathrm{Ad}_h X) = d(\sigma C_h)X = d(C_h\sigma)X = \mathrm{Ad}_h(\sigma X) = \mathrm{Ad}_h(-X) = -\mathrm{Ad}_h X.$$

Finally, $H$ is a Lie subgroup so $\mathfrak{h}$ is closed under the bracket operation. The second containment in (4) holds since $\mathfrak{p}$ is $\mathrm{Ad}(H)$-invariant. For the third, simply notice that if $X, Y \in \mathfrak{p}$ then $d\sigma([X, Y]) = [d\sigma X, d\sigma Y] = [-X, -Y] = [X, Y]$ so $[X, Y] \in \mathfrak{h}$. [O, p.316] □

**Remark A.** A $G$-invariant metric always exists for a symmetric pair $(G, H)$. Since $\mathrm{Ad}_G H$ is compact in the topology of $GL(\mathfrak{g})$, there exists a positive definite quadratic form $B$ on $\mathfrak{p}$ invariant under $\mathrm{Ad}_G H$. Then $Q = B \circ (d\pi)^{-1}$ on $T_pM$ is invariant under $d\tau_h$ for any $h \in H$. Let $Q$ also denote the symmetric bilinear form. The metric on $G/H$ is then given by

$$\langle v, w \rangle_q = Q(d\tau_{g^{-1}}v, d\tau_{g^{-1}}w)$$

where $q = gH$ in $G/H$ and $v, w$ are in $T_qM$. [He, p.210]

**Remark B.** Let $(G, H)$ be a symmetric pair and $\mathfrak{z}$ the Lie algebra of the center of $G$. If $\mathfrak{h} \cap \mathfrak{z} = \{0\}$ then there exists exactly one involutive automorphism $\sigma$ of $G$ such that $F_0^\sigma \subset H \subset F^\sigma$. [He, p.212]

**Remark C.** The map $d\pi$ allows us to identify the vector subspace $\mathfrak{p}$ of $\mathfrak{g}$ with the tangent space $T_pM$ at $p = \pi(e)$.

We are now ready to prove Theorem 1.5.

*Proof.* The relation $S \circ \pi = \pi \circ \sigma$ defines the function $S : M \to M$. For if $\pi g_1 = \pi g_2$ then $g_1 H = g_2 H$ and $\sigma(g_1)H = \sigma(g_2)H$ since $H \subset F^\sigma$. Thus, $\pi\sigma(g_1) = \pi\sigma(g_2)$. As $\sigma$ is an involution it follows that $S$ is an involution and $S(p) = S(\pi(e)) = \pi\sigma(e) = \pi(e) = p$.

For $y \in T_pM$, find $Y$ in $\mathfrak{g}$ so that $d\sigma(Y) = -Y$ and $d\pi(Y) = y$. Then

$$dS_p(y) = dS_p d\pi(Y) = d\pi d\sigma(Y) = d\pi(-Y) = -y.$$

That is, $dS_p = -I$ on $T_pM$. Now, let $g \in G$. For any $a$ in $G$ we have

$$S\tau_g(\pi a) = S\pi(ga) = \pi\sigma(ga) = \pi(\sigma g \cdot \sigma a) = \tau_{\sigma g}\pi(\sigma a) = \tau_{\sigma g}S(\pi a)$$

and so $\tau_{\sigma g} = S\tau_g S$ for all $g \in G$.

Suppose that $\langle , \rangle$ is any $G$-invariant metric on $M$. We wish to show that $S$ is an isometry. Take $q = gH$ in $M$ and consider $v \in T_qM$. Put $w = d\tau_{g^{-1}}(v) \in T_pM$.

$$\begin{aligned}
\langle dS_q(v), dS_q(v)\rangle &= \langle dS_q d\tau_g d\tau_{g^{-1}}(v), dS_q d\tau_g d\tau_{g^{-1}}(v)\rangle \\
&= \langle d\tau_{\sigma g} dS_p(w), d\tau_{\sigma g} dS_p(w)\rangle \\
&= \langle dS_p(w), dS_p(w)\rangle \\
&= \langle -w, -w\rangle \\
&= \langle v, v\rangle.
\end{aligned}$$

Finally, we observe that if any homogeneous space $G/H$ has a symmetry $S$ about a single point $p$ then the symmetry $S^q$ about any other point $q$ can be defined by $S^q = g \circ S \circ g^{-1}$ where $g$ is any isometry such that $g(p) = q$. [O, p.315] □

The representation of a symmetric space as $G/H$ and the study of the Lie algebra $\mathfrak{g}$ yields some important geometric information about the manifold. We see in particular that the curvature tensor $R$ is independent of the choice of $G$-invariant metric.

**Proposition 1.7.** *The geodesics starting at* $p = \pi(e)$ *are given by*

$$\gamma_{d\pi X}(t) = \alpha(t)p = \pi\alpha(t)$$

*where* $\alpha$ *is the one parameter subgroup generated by* $X \in \mathfrak{p}$. *The curvature tensor at* $p$ *is given by*

$$R(X,Y)Z = [Z, [X,Y]]$$

*where we have identified* $T_pM$ *with* $\mathfrak{p}$. *If* $X$ *and* $Y$ *span a nondegenerate plane in* $T_pM$, *then the sectional curvature can be computed as*

$$K(X,Y) = \frac{\langle [[X,Y],X],Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

A complete proof can be found in [O, p.317]. Here we compute the curvature tensor following Petersen. It should be noted that Petersen's approach is slightly different than the one we have taken thus far. He identifies the Lie algebra $\mathfrak{g}$ of $I_0(M)$ with the algebra $\mathfrak{k}$ of Killing fields. A Killing field on a Riemannian manifold $(M, g)$ is a vector field $X$ for which $L_X g \equiv 0$, which holds if and only if $v \mapsto \nabla_v X$ is a skew-symmetric (1,1)-tensor. The Killing field $X$ is uniquely determined by the pair $(X_p, (\nabla X)_p)$ at some point $p$ [P, p.164]. If $M$ is a symmetric space, we obtain the identification $\mathfrak{k} = T_pM \times \mathfrak{h} \cong \mathfrak{g}$ [P, p.213]. For a general Riemannian manifold, there is only an (anti-)isomorphism of $\mathfrak{g}$ onto the subalgebra of complete Killing fields in $\mathfrak{k}$ [O, p.255].

*Proof.* We aim to show that for $X, Y, Z \in \mathfrak{p}$ we have $R(X,Y)Z = [Z, [X,Y]]$. Each of the vectors $X, Y, Z \in T_pM$ determine a Killing field $X, Y, Z$ on $M$ with $\nabla X = \nabla Y = \nabla Z = 0$ at point $p$. We first show that for any Killing field $K$ on a Riemannian manifold,

$$\nabla^2_{X,Y} K = -R(K, X)Y$$

where the second covariant derivative is $\nabla^2_{X,Y} K = \nabla_X \nabla_Y K - \nabla_{\nabla_X Y} K$. In this notation we have that

$$R(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

As $K$ is a Killing field we know that $v \mapsto \nabla_v K$ is a skew-symmetric $(1,1)$-tensor and so $Y \mapsto \nabla^2_{X,Y} K$ is also skew-adjoint [P, p.167]. For any vector field $Z$,

$$\begin{aligned}
\langle \nabla^2_{X,Y} K, Z\rangle &= -\langle \nabla^2_{X,Z} K, Y\rangle \\
&= -\langle \nabla^2_{Z,X} K, Y\rangle - \langle R(X,Z)K, Y\rangle \\
&= \langle \nabla^2_{Z,Y} K, X\rangle - \langle R(X,Z)K, Y\rangle \\
&= \langle \nabla^2_{Y,Z} K, X\rangle + \langle R(Z,Y)K, X\rangle - \langle R(X,Z)K, Y\rangle \\
&= -\langle \nabla^2_{Y,X} K, Z\rangle + \langle R(Z,Y)K, X\rangle - \langle R(X,Z)K, Y\rangle \\
&= -\langle \nabla^2_{X,Y} K, Z\rangle - \langle R(Y,X)K, Z\rangle + \langle R(Z,Y)K, X\rangle \\
&\quad - \langle R(X,Z)K, Y\rangle
\end{aligned}$$

and so $2\langle \nabla^2_{X,Y} K, Z\rangle = -\langle R(Y,X)K, Z\rangle + \langle R(Z,Y)K, X\rangle - \langle R(X,Z)K, Y\rangle$. From Bianchi's first identity and the symmetries of $R$,

$$\begin{aligned}
&-\langle R(Y,X)K, Z\rangle + \langle R(Z,Y)K, X\rangle - \langle R(X,Z)K, Y\rangle \\
&\qquad = -\langle R(K,X)Y, Z\rangle + \langle R(Y,K)X, Z\rangle + \langle R(X,Y)K, Z\rangle \\
&\qquad = -2\langle R(K,X)Y, Z\rangle
\end{aligned}$$

and thus $2\langle \nabla^2_{X,Y} K, Z\rangle = -2\langle R(K,X)Y, Z\rangle$.

Now, since the covariant derivatives of $X, Y, Z$ all vanish at $p$, we have

$$\begin{aligned} R(X,Y)Z &= R(X,Z)Y - R(Y,Z)X \\ &= -\nabla_Z\nabla_Y X + \nabla_Z \nabla_X Y \\ &= \nabla_Z[X,Y] \\ &= [Z,[X,Y]] \end{aligned}$$

[P, p.215] □

**Example**

Here we show that the real Grassmannians (which include the spheres and real projective space as special cases) are symmetric spaces. We write the manifold as $G/H$, display the symmetry at a point, and show the decomposition of the Lie algebra. Finally we compute the sectional curvature using Proposition 1.7.

Let $G_{pq}$ denote the Grassmannian of unoriented p-planes in $\mathbf{R}^{p+q}$. We may write $G_{pq}$ as

$$\begin{aligned} G_{pq} &= \mathcal{O}(p+q)/\mathcal{O}(p) \times \mathcal{O}(q) \\ &= \mathcal{SO}(p+q)/\mathcal{S}(\mathcal{O}(p) \times \mathcal{O}(q)). \end{aligned}$$

The simply connected Grassmannian $\tilde{G}_{pq}$ of oriented p-planes in $\mathbf{R}^{p+q}$ is a double cover of $G_{pq}$ and has a presentation as

$$\tilde{G}_{pq} = \mathcal{SO}(p+q)/\mathcal{SO}(p) \times \mathcal{SO}(q).$$

We note that $\tilde{G}_{1n} = S^n$ and $G_{1n} = \mathbf{R}P^n$. Let us concentrate on $\tilde{G}_{pq}$. Fix the standard basis $\{e_1, \dots, e_{p+q}\}$ for $\mathbf{R}^{p+q}$. Let point $P$ be the span of $\{e_1 \dots, e_p\}$ and define the symmetry at $P$ by

$$S_p = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \in \mathcal{O}(p+q).$$

The involution $\sigma$ on $G = \mathcal{SO}(p+q)$ is defined by $\sigma(A) = S_p A S_p$ so

$$\sigma \begin{pmatrix} B & C \\ D & E \end{pmatrix} = \begin{pmatrix} B & -C \\ -D & E \end{pmatrix}.$$

Clearly the fixed point set is $F^\sigma = S(\mathcal{O}(p) \times \mathcal{O}(q))$ and thus letting $H$ be the identity component of $F^\sigma$, we have $H = S\mathcal{O}(p) \times S\mathcal{O}(q)$. The Lie algebra is $\mathfrak{g} = \mathfrak{o}(p+q)$, the space of skew-symmetric transformations. As $S_p = S_p^{-1}$, the action of $\sigma$ is by conjugation and thus $d\sigma$ is also action by conjugation on $\mathfrak{g}$. We find

$$\begin{aligned} \mathfrak{h} &= \{X \in \mathfrak{g} : d\sigma(X) = X\} \\ &= \left\{ \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix} : W, Z \text{ are skew-symmetric} \right\} \\ &= \mathfrak{o}(p) \times \mathfrak{o}(q) \end{aligned}$$

$$\begin{aligned}
\mathfrak{p} &= \{X \in \mathfrak{g} : d\sigma(X) = -X\} \\
&= \left\{ \begin{pmatrix} 0 & -C_X^t \\ C_X & 0 \end{pmatrix} : C_X \in M_{p,q}(\mathbf{R}) \right\} \\
&= \mathbf{R}^{pq}
\end{aligned}$$

We can take the standard Euclidean metric on $M_{p,q}(\mathbf{R})$ given by $\langle A, B\rangle = \operatorname{Tr} AB^t$. This gives a form on $\mathfrak{p}$

$$B(X,Y) = \operatorname{Tr}(C_X C_Y^t) = \frac{1}{2}\operatorname{Tr}(XY^t)$$

which is $\operatorname{Ad} H$ invariant. The action of $\operatorname{Ad} H$ on $\mathfrak{p}$ is as

$$\begin{aligned}
\operatorname{Ad}(h)X &= hXh^{-1} \\
&= \begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} 0 & -C_X^t \\ C_X & 0 \end{pmatrix} \begin{pmatrix} H^{-1} & 0 \\ 0 & K^{-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & -HC_X^t K^{-1} \\ KC_X H^{-1} & 0 \end{pmatrix} \in \mathfrak{p}
\end{aligned}$$

where $-(KC_XH^{-1})^t = -HC_X^tK^{-1}$ since $H^t = H^{-1}$ and $K^t = K^{-1}$. Having identified the tangent space $T_P\tilde{G}_{pq}$ with $\mathfrak{p}$ via $d\pi$ we wish to compute the sectional curvature using $R(X,Y)Z = [Z,[X,Y]]$. For notational simplicity, we write $X$ for $C_X$,$Y$ for $C_Y$, $Z$ for $C_Z$.

$$\begin{aligned}
[X,Y] &= \left[ \begin{pmatrix} 0 & -X^t \\ X & 0 \end{pmatrix}, \begin{pmatrix} 0 & -Y^t \\ Y & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} Y^tX - X^tY & 0 \\ 0 & YX^t - XY^t \end{pmatrix}
\end{aligned}$$

and we compute

$$[[X,Y],Z] = \begin{pmatrix} 0 & \begin{matrix} X^tYZ^t - Y^tXZ^t \\ + Z^tYX^t - Z^tXY^t \end{matrix} \\ \begin{matrix} YX^tZ - XY^tZ \\ - ZY^tX + ZX^tY \end{matrix} & 0 \end{pmatrix}$$

which lies in $\mathfrak{p}$. We compute

$$\begin{aligned}
\langle [[X,Y],X],Y\rangle &= \frac{1}{2}\operatorname{Tr}((YX^tX - XY^tX - XY^tX + XX^tY)Y^t) \\
&\quad + \frac{1}{2}\operatorname{Tr}((X^tXY^t - X^tYX^t - X^tYX^t + Y^tXX^t)Y) \\
&= \frac{1}{2}\big(\langle YX^t, YX^t\rangle - 2\langle XY^t, YX^t\rangle + \langle X^tY, X^tY\rangle \\
&\quad + \langle XY^t, XY^t\rangle - 2\langle X^tY, Y^tX\rangle + \langle Y^tX, Y^tX\rangle\big) \\
&= \frac{1}{2}\big(\|XY^t - YX^t\|^2 + \|X^tY - Y^tX\|^2\big) \\
&\geq 0
\end{aligned}$$

[P, p.220]. The sectional curvature $K(X,Y)$ given by the formula in Proposition 1.7 then satisfies $K \geq 0$.

Let us consider now the special case of the sphere $S^n = \tilde{G}_{1n}$. The computations simplify as follows. We identify $X \in \mathfrak{p}$ with a vector $x \in \mathbf{R}^n$ and $B(X,Y) = x \cdot y$, th standard inner product on $\mathbf{R}^n$. With this inner product we obtain the usual metric on $S^n$. We compute

$$[[X,Y],Z] = \begin{pmatrix} 0 & -v^t \\ v & 0 \end{pmatrix}$$

where $v = (x \cdot z)y - (y \cdot z)x$. Therefore,

$$\begin{aligned} K(X,Y) &= \frac{\langle [[X,Y],X],Y\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2} \\ &= \frac{|x|^2|y|^2 - (y \cdot x)(x \cdot y)}{|x|^2|y|^2 - (x \cdot y)^2} \\ &\equiv 1 \end{aligned}$$

Now, Proposition 1.7 also describes the geodesics of a symmetric space: for the sphere $S^n$ with point $P = (1,0,\ldots,0)$, we have $\gamma_X(t) = (\exp tX)P, X \in \mathfrak{p}$. A direct computation of the power series for $\exp tX$ shows that $\gamma_u(t) = (\cos t)P + (\sin t)u$ where $u$ is a unit vector in $\mathbf{R}^n = \mathfrak{p}$. Thus the geodesics through $P$ are the great circles. [O, p.318]

The interesting example of complex projective space $M = \mathbf{C}P^n$ as a complex Grassmannian is worked out carefully in [P, p.222]. Let $P \in M$ be the first coordinate axis in $\mathbf{C}^{n+1}$ and $z, w \in T_P M$ unit vectors with $\operatorname{Re}\langle z,w\rangle = 0$; the inner product is the usual one on $\mathbf{C}^n$. The sectional curvature is then computed to be $K(z,w) = 1 + 3|\operatorname{Im}\langle z,w\rangle|^2$. If $z$ and $w$ are orthogonal then the sectional curvature is 1, but if $w = iz$ then $K(z,w) = 4$. All other curvatures fall between these two values so complex projective space is "quarter pinched".

## 2. A Review of Lie algebras

We begin this section with a discussion of the basics of Lie algebra theory [S, Ch.I,II] and the structure of complex semisimple Lie algebras [S], [Hu], [He]. A definition of compact and noncompact Lie algebras is given and the Cartan decomposition is described [He, Ch.III]. No proofs will be given.

### The basics

We let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $k$ of characteristic 0. A **lower central series** is a descending series of ideals of $\mathfrak{g}$ given by

$$\mathfrak{g} \supset [\mathfrak{g},\mathfrak{g}] \supset [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]] \supset [\mathfrak{g},[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]] \supset \cdots$$

The Lie algebra is said to be **nilpotent** if this series terminates at 0. If $[\mathfrak{g},\mathfrak{g}] = 0$ then $\mathfrak{g}$ is **abelian**.

**2.1.** *A Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if* $\operatorname{ad} X$ *is a nilpotent transformation of* $\mathfrak{g}$ *for each* $X$ *in* $\mathfrak{g}$.

The **derived series** of a Lie algebra is given by

$$\mathfrak{g} \supset [\mathfrak{g}, \mathfrak{g}] \supset [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supset \cdots$$

and $\mathfrak{g}$ is said to be **solvable** if this series terminates at 0. We note that any nilpotent algebra is solvable. There is a largest solvable ideal $\mathfrak{r}$ of $\mathfrak{g}$ called the **radical** of $\mathfrak{g}$. We say $\mathfrak{g}$ is **semisimple** if the radical is 0. For any Lie algebra, $\mathfrak{g}/\mathfrak{r}$ is semisimple. Define a symmetric bilinear form $B$, the **Killing form**, on $\mathfrak{g}$ by $B(X, Y) = \operatorname{Tr}(\operatorname{ad} X \operatorname{ad} Y)$. Note that $B$ satisfies $B([X, Y], Z) = B(Y, [X, Z])$.

**2.2.** *If* $\mathfrak{a}$ *is an ideal of* $\mathfrak{g}$ *and* $\mathfrak{a}'$ *is orthogonal to* $\mathfrak{a}$ *with respect to* $B$ *then* $\mathfrak{a}'$ *is also an ideal of* $\mathfrak{g}$.

**2.3.** *A Lie algebra is semisimple if and only if its Killing form is nondegenerate.*

A Lie algebra is said to be **simple** if it is not abelian and contains no nontrivial ideals.

**2.4.** *A Lie algebra is semisimple if and only if it is isomorphic to a product of simple algebras.*

**2.5.** *If* $\mathfrak{g}$ *is semisimple then* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

A **derivation** of $\mathfrak{g}$ is a linear mapping $D$ such that $D([X, Y]) = [DX, Y] + [X, DY]$. A derivation is **inner** if it is of the form $\operatorname{ad} X$ for some $X$ in $\mathfrak{g}$.

**2.6.** *Every derivation of a semisimple Lie algebra is inner and we obtain an isomorphism* $\operatorname{ad} : \mathfrak{g} \to \mathfrak{D}(\mathfrak{g}) =$ *derivations of* $\mathfrak{g}$.

**2.7.** *If* $G$ *is a connected Lie group (real or complex) with semisimple Lie algebra, then the identity component* $Aut_0(G)$ *of the group of automorphisms of* $G$ *coincides with the inner automorphisms of* $G$.

The complex semisimple Lie algebras have been completely classified as follows. There are four series $A_n$, $B_n$, $C_n$, $D_n$ defined by

- For $n \geq 1$, $A_n = \mathfrak{sl}(n+1)$ the Lie algebra of $SL(n+1)$.
- For $n \geq 2$, $B_n = \mathfrak{so}(2n+1)$ the Lie algebra of $SO(2n+1)$.
- For $n \geq 3$, $C_n = \mathfrak{sp}(2n)$ the Lie algebra of $Sp(2n)$.
- For $n \geq 4$, $D_n = \mathfrak{so}(2n)$ the Lie algebra of $SO(2n)$.

In addition to these, there are five "exceptional" simple Lie algebras over $\mathbf{C}$ denoted by $G_2, F_4, E_6, E_7$ and $E_8$ of dimensions 14, 52, 78, 133, and 248, respectively. [S, p.8]

## Cartan subalgebras and the Root Space Decomposition

Here we let $\mathfrak{g}$ be a Lie algebra over $\mathbf{C}$. A **Cartan subalgebra** $\mathfrak{t}$ of $\mathfrak{g}$ is one such that (i) $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and (ii) for each $X \in \mathfrak{t}$, $\operatorname{ad} X$ is a semisimple element (diagonalizable).

**2.8.** *Every semisimple Lie algebra over* $\mathbf{C}$ *contains a Cartan subalgebra.*

**2.9.** *The group* $Int(\mathfrak{g})$ *of inner automorphisms of* $\mathfrak{g}$ *acts transitively on the set of Cartan subalgebras of* $\mathfrak{g}$.

Now fix a Cartan subalgebra $\mathfrak{t}$ of semisimple $\mathfrak{g}$. Since $\mathfrak{t}$ is abelian, $\operatorname{ad}_{\mathfrak{g}} \mathfrak{t}$ is a commuting family of endomorphisms of $\mathfrak{g}$ and thus simultaneously diagonalizable. We may then write $\mathfrak{g}$ as a direct sum of subspaces

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [h, X] = \alpha(h)X \text{ for all } h \in \mathfrak{t}\}$$

where $\alpha$ lies in the dual of $\mathfrak{t}$. We note that $\mathfrak{g}^0$ is the centralizer of $\mathfrak{t}$. The set of all non-zero $\alpha \in \mathfrak{t}^*$ for which $\mathfrak{g}^\alpha \neq \{0\}$ will be denoted by $\Delta$. The elements of $\Delta \cup \{0\}$ are called the **roots** of $\mathfrak{g}$ relative to $\mathfrak{t}$. Since $\mathfrak{t}$ is maximal abelian we have $\mathfrak{t} = \mathfrak{g}^0$, and the Jacobi identity gives $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$. We thus obtain the **root space decomposition**

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha.$$

**2.10.** *The root space decomposition has the following properties.*

(1) *The dimension of* $\mathfrak{g}^\alpha$ *is 1 for each* $\alpha \in \Delta$.
(2) *If* $\alpha$ *and* $\beta$ *are roots with* $\alpha + \beta \neq \{0\}$ *then* $\mathfrak{g}^\alpha$ *and* $\mathfrak{g}^\beta$ *are orthogonal with respect to the Killing form* $B$.
(3) *The restriction of* $B$ *to* $\mathfrak{t}$ *is nondegenerate. For each* $\alpha \in \mathfrak{t}^*$ *there exists a unique element* $h_\alpha \in \mathfrak{t}$ *so that* $B(h, h_\alpha) = \alpha(h)$ *for all* $h \in \mathfrak{t}$. *We define* $\langle \alpha, \beta \rangle = B(h_\alpha, h_\beta)$.
(4) *If* $\alpha \in \Delta$ *then* $-\alpha \in \Delta$ *and* $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathbf{C} \cdot h_\alpha$, *the* $\mathbf{C}$*-span of* $h_\alpha$.
(5) *The only roots proportional to* $\alpha$ *are* $-\alpha, 0, \alpha$.

**2.11.** *Let* $\mathfrak{t}_R = \bigoplus_{\alpha \in \Delta} \mathbf{R} \cdot h_\alpha$. *Then* $B$ *is real and strictly positive definite on* $\mathfrak{t}_R$ *and* $\mathfrak{t} = \mathfrak{t}_R \oplus i\mathfrak{t}_R$.

A semisimple Lie algebra over $\mathbf{C}$ is determined (up to isomorphism) by a Cartan subalgebra and the pattern of roots.

**2.12.** *Let* $\mathfrak{g}, \mathfrak{g}'$ *be semisimple Lie algebras and* $\mathfrak{t}, \mathfrak{t}'$ *Cartan subalgebras. Let* $\Delta, \Delta'$ *be the non-zero roots and define* $\mathfrak{t}_R = \bigoplus_{\alpha \in \Delta} \mathbf{R} \cdot h_\alpha$ *and similarly define* $\mathfrak{t}'_R$. *Then* $\Delta$ *and* $\Delta'$ *may be considered as subsets of the dual spaces to* $\mathfrak{t}_R$ *and* $\mathfrak{t}'_R$ *respectively, since each* $\beta \in \Delta$ $(\Delta')$ *is real on* $\mathfrak{t}_R$ $(\mathfrak{t}'_R)$. *If* $\phi$ *is an injective* $\mathbf{R}$*-linear mapping of* $\mathfrak{t}_R$ *onto* $\mathfrak{t}'_R$ *such that* $\phi^t$ *maps* $\Delta'$ *onto* $\Delta$, *then* $\phi$ *extends to an isomorphism of* $\mathfrak{g}$ *onto* $\mathfrak{g}'$.

## Ordering the roots

Let $V$ be a real vector space and fix a basis $X_1, \ldots, X_n$. We order $V$ as follows. For $X, Y \in V$ write $X - Y = \sum a_i X_i$. We say $X > Y$ if the first non-zero term in the sequence $a_1 \ldots a_n$ is positive. For $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, we write $\mathfrak{t}_R = \sum_{\alpha \in \Delta} \mathbf{R}\mathfrak{t}_\alpha$. $\Delta$ is a subset of the dual space of $\mathfrak{t}_R$ and can be ordered with respect to the ordering of $\mathfrak{t}_R^*$ for some fixed basis. [He, p.173]

Let $V^*$ be the dual of $V$ and $W$ a subspace of $V$. Given an ordering on $V$ we order the dual by declaring $\lambda \in V^*$ positive if the first non-zero term in $\lambda(X_1), \ldots, \lambda(X_n)$ is positive. The orderings on $V$ and $W$ are **compatible** if an element $\lambda \in V^*$ is positive whenever the restriction to $W$ is positive. Compatible orderings can always be constructed by starting with a basis $w_1, \ldots, w_m$ of $W$ and extending to a basis $w_1, \ldots, w_m, w_{m+1}, \ldots, w_n$ for $V$. [He, p.260]

### More notation and definitions

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$. We define the following.

- $GL(\mathfrak{g})$ = Lie group of nonsingular linear transformations
- $\mathfrak{gl}(\mathfrak{g})$ = algebra of all endomorphisms of $\mathfrak{g}$, $[A, B] = AB - BA$
- $Int(\mathfrak{g})$ = the subgroup of $GL(\mathfrak{g})$ with Lie algebra $\operatorname{ad}\mathfrak{g}$
- $Aut(\mathfrak{g})$ = the closed subgroup of automorphism of $\mathfrak{g}$ in $GL(\mathfrak{g})$
- $\mathfrak{D}(\mathfrak{g})$ = the Lie algebra of $Aut(\mathfrak{g})$, the derivations of $\mathfrak{g}$.

For any Lie algebra $\mathfrak{g}$, we have $\operatorname{ad}\mathfrak{g} \subset \mathfrak{D}(\mathfrak{g})$ and $Int(\mathfrak{g}) \subset Aut(\mathfrak{g})$. The map $X \mapsto \operatorname{ad} X$ is a homomorphism of $\mathfrak{g}$ onto $\operatorname{ad}\mathfrak{g}$ with kernel the center of $\mathfrak{g}$. A semisimple Lie algebra $\mathfrak{g}$ is isomorphic to $\operatorname{ad}\mathfrak{g}$ and we have $\operatorname{ad}\mathfrak{g} = \mathfrak{D}(\mathfrak{g})$, $Int(\mathfrak{g}) = Aut_0(\mathfrak{g})$, the identity component of $Aut(\mathfrak{g})$.

Returning to the general Lie algebra $\mathfrak{g}$ over $\mathbf{R}$, let $\mathfrak{h}$ be a subalgebra and let $K$ denote the subgroup of $Int(\mathfrak{g})$ with Lie subalgebra $\operatorname{ad}_{\mathfrak{g}}\mathfrak{h}$ of $\operatorname{ad}_{\mathfrak{g}}\mathfrak{g}$. We say $\mathfrak{h}$ is **compactly imbedded** in $\mathfrak{g}$ if $K$ is compact. We say $\mathfrak{g}$ is **compact** if $Int(\mathfrak{g})$ is compact. We say a Lie algebra is **noncompact** if it is semisimple and contains no compact ideals.

**2.13.** *For semisimple $\mathfrak{g}$ over $\mathbf{R}$, $\mathfrak{g}$ is compact if and only if its Killing form is strictly negative definite.*

**2.14.** *A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is compact if and only if there exists a compact Lie group $G$ with Lie algebra isomorphic to $\mathfrak{g}$.*

**2.15.** *Suppose Lie algebra $\mathfrak{g}$ is a direct sum of two ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$. Let $\mathfrak{h}_1, \mathfrak{h}_2$ be subalgebras of $\mathfrak{g}_1, \mathfrak{g}_2$ and set $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. Then $\mathfrak{h}$ is a compactly imbedded subalgebra if and only if $\mathfrak{h}_1, \mathfrak{h}_2$ are compactly imbedded in $\mathfrak{g}_1, \mathfrak{g}_2$ respectively.* [He, p.234]

Let $G$ be a Lie group and $H$ a closed subgroup and consider $M = G/H$. Denote by $\tau_g$ the map $xH \mapsto gxH$ where $g \in G$. If $p$ is the point $\pi(e)$, the group of all linear transformations $(d\tau_h)_p$ on $T_pM$ for $h \in H$ is called the **linear isotropy group** and denoted by $H^*$.

A Lie algebra $\mathfrak{g}$ over $\mathbf{R}$ is said to have a complex structure $J$ if $J$ is an $\mathbf{R}$-linear endomorphism of the vector space $\mathfrak{g}$ such that $J^2 = -I$ and $[X, JY] = J[X, Y]$. Such a Lie algebra can be considered as a Lie algebra over $\mathbf{C}$ by setting $(a + ib)X = aX + bJX$. On the other hand, if $\mathfrak{g}$ is a Lie algebra over $\mathbf{C}$, we consider a real Lie algebra $\mathfrak{g}^R$ where the multiplication by $i$ becomes a complex structure and $\dim_{\mathbf{R}} \mathfrak{g}^R = 2 \dim_{\mathbf{C}} \mathfrak{g}$.

If $\mathfrak{g}$ is a Lie algebra over $\mathbf{R}$, we define the **complexification** $\mathfrak{g}^C = \mathfrak{g} \otimes \mathbf{C}$ and so $\dim_{\mathbf{C}} \mathfrak{g}^C = \dim_{\mathbf{R}} \mathfrak{g}$. This $\mathfrak{g}^C$ could also be described as the product $\mathfrak{g} \times \mathfrak{g}$ with complex structure given by $J : (X, Y) \mapsto (-Y, X)$. We write the pair $(X, Y)$ as $X + iY$.

**2.16.** *The Lie algebras $\mathfrak{g}$ over $\mathbf{R}$, $\mathfrak{g}^C$, and $(\mathfrak{g}^C)^R$ are all semisimple if and only if one of them is.*

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{C}$. A **real form** of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real algebra $\mathfrak{g}^R$ such that $\mathfrak{g}^R = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$. If such a real form exists, then $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$ and we write elements in $\mathfrak{g}$ uniquely as $X + iY$ for $X, Y \in \mathfrak{g}_0$. A map $\sigma : X + iY \mapsto X - iY$ is known as a **conjugation** of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$.

**2.17.** *Every semisimple Lie algebra over $\mathbf{C}$ has a real form which is compact.*

## Cartan Decomposition

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$ and $\mathfrak{g}^C$ its complexification. Put $\sigma$ as the conjugation with respect to $\mathfrak{g}$. A direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ into a subalgebra $\mathfrak{h}$ and a vector space $\mathfrak{p}$ is a **Cartan decomposition** if there exists a compact real form $\mathfrak{g}_0$ of $\mathfrak{g}^C$ such that $\sigma(\mathfrak{g}_0) \subset \mathfrak{g}_0$, $\mathfrak{h} = \mathfrak{g} \cap \mathfrak{g}_0$, and $\mathfrak{p} = \mathfrak{g} \cap (i\mathfrak{g}_0)$.

**2.18.** *Every semisimple Lie algebra over $\mathbf{R}$ has a Cartan decomposition.*

**2.19.** *Any two Cartan decompositions are conjugate under an inner automorphism. That is, if $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{p}_1 = \mathfrak{h}_2 \oplus \mathfrak{p}_2$ are Cartan decompositions, then there exists a $\psi \in Int(\mathfrak{g})$ such that $\psi(\mathfrak{h}_1) \subset \mathfrak{h}_2$ and $\psi(\mathfrak{p}_1) \subset \mathfrak{p}_2$.*

**2.20.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbf{R}$ which is a direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ is a subalgebra and $\mathfrak{p}$ is a vector space. The following are equivalent:*

(1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ *is a Cartan decomposition.*
(2) *the map $s : T + X \mapsto T - X$ with respect to this decomposition is an automorphism and the symmetric bilinear form*

$$B_s(X, Y) = -B(X, sY)$$

*is strictly positive definite. That is, the Killing form $B$ is negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{p}$.*

*If these conditions are satisfied, then $\mathfrak{h}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}$.*

If $\mathfrak{g}$ is semisimple over $\mathbf{R}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a Cartan decomposition, then $\mathfrak{u} = \mathfrak{h} + i\mathfrak{p}$ is a compact real form of the complexification $\mathfrak{g}^C$.

**2.21.** *Suppose $\mathfrak{g}$ is semisimple over $\mathbf{C}$ and $\mathfrak{u}$ is a compact real form of $\mathfrak{g}$. Let $J$ denote the complex structure on $\mathfrak{g}^R$ which corresponds to multiplication by $i$ on $\mathfrak{g}$. Then $\mathfrak{g}^R = \mathfrak{u} \oplus J\mathfrak{u}$ is a Cartan decomposition of $\mathfrak{g}^R$.*

## 3. Symmetric Lie Algebras

A pair $(\mathfrak{g}, s)$ is known as a **symmetric Lie algebra** if

(i) $\mathfrak{g}$ is a Lie algebra over $\mathbf{R}$,
(ii) $s$ is an involutive automorphism of $\mathfrak{g}$, and
(iii) $\mathfrak{h}$, the set of fixed points of $s$, is a compactly imbedded subalgebra of $\mathfrak{g}$.

The pair is called **effective** if in addition

(iv) $\mathfrak{h} \cap \mathfrak{z} = \{0\}$ where $\mathfrak{z}$ is the center of $\mathfrak{g}$.

A pair $(G, H)$ where $G$ is a connected Lie group and $H$ is a Lie subgroup is said to be **associated** to the pair $(\mathfrak{g}, s)$ if the Lie algebras of $G$ and $H$ are $\mathfrak{g}$ and $\mathfrak{h}$. Two symmetric Lie algebras $(\mathfrak{g}_1, s_1)$ and $(\mathfrak{g}_2, s_2)$ are said to be isomorphic if there exists an isomorphism $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $\phi \circ s_1 = s_2 \circ \phi$.

**Proposition 3.1.** *Any symmetric space $M = G/H$ gives rise to an effective symmetric Lie algebra.*

*Proof.* That the pair $(\mathfrak{g}, d\sigma)$ is a symmetric Lie algebra was established in Sectin 1. It remains to see that the pair is effective. The group $I(M)$ acts effectively on $G/H$ where $H$ is the isotropy group of a point in $M$ (that is, no nontrivial element in $I(M)$ leaves every point of $M$ fixed). Therefore, $H$ cannot contain any normal subgroup $\neq \{0\}$ of $G$ [He, p.208]. This gives proplerty (iv). □

**Theorem 3.2.** *Let $(\mathfrak{g}, s)$ be an effective symmetric Lie algebra and $\mathfrak{h}$ the fixed point set of $s$. Let $(G, H)$ and $(\tilde{G}, \tilde{H})$ be two pairs associated to $(\mathfrak{g}, s)$ with $\tilde{G}$ simply connected and $H, \tilde{H}$ connected. Then $\tilde{H}$ is closed in $\tilde{G}$ and $(\tilde{G}, \tilde{H})$ is a symmetric pair. If $H$ is closed in $G$, then $G/H$ is a locally symmetric space for each $G$-invariant metric, and $\tilde{G}/\tilde{H}$ is the universal cover of $G/H$.*

*Proof.* Since $\tilde{G}$ is simply connected, there exists a smooth homomorphism $\sigma : \tilde{G} \to \tilde{G}$ so that $d\sigma_e = s$. This $\sigma$ is then an involutive automorphism and $\tilde{H}$ is the identity component of the group of fixed points of $\sigma$. In particular, $\tilde{H}$ is closed in $\tilde{G}$. To show that $\tilde{G}/\tilde{H}$ is simply connected, let $\eta : [0, 1] \to \tilde{G}/\tilde{H}$ be a loop so that $\eta(0) = \eta(1) = \tilde{\pi}(e)$ where $\tilde{\pi}$ is the projection mapping $\tilde{G} \to \tilde{G}/\tilde{H}$. Find a lift $\tilde{\eta} : [0, 1] \to \tilde{G}$ of $\eta$ so that

$\tilde{\eta}(0) = e$. Since $\tilde{H}$ is connected we can find a path $\omega$ lying in $\tilde{H}$ joining $\tilde{\eta}(1)$ to $\tilde{\eta}(0)$. The closed loop $\tilde{\eta} \cdot \omega$ is null-homotopic in $\tilde{G}$ which implies that $\eta$ is null-homotopic in $\tilde{G}/\tilde{H}$.

Now, let us note that $\mathrm{Ad}_G H$ and $\mathrm{Ad}_{\tilde{G}} \tilde{H}$ coincide in the group $Int(\mathfrak{g})$ since they have the same Lie algebra. These are compact since $\mathfrak{h}$ is compactly imbedded in $\mathfrak{g}$. Note that $\mathfrak{p} = \{X \in \mathfrak{g} : s(X) = -X$ is invariant under the action of $\mathrm{Ad}_G H$ and carries a strictly positive definite quadratic form $Q$ invariant under $\mathrm{Ad}_G H$. As in Remark A of Section 1, this gives a $\tilde{G}$-invariant Riemannian metric on $\tilde{G}/\tilde{H}$ and if $H$ is closed in $G$, also a $G$-invariant Riemannian metric on $G/H$. Finally let $\phi$ be the homomorphism of $\tilde{G}$ onto $G$ so that $d\phi_e$ is the identity on $\mathfrak{g}$. Then $\tilde{H}$ is the identity component of $\phi^{-1}(H)$ and $\tilde{G}/\tilde{H} \to G/\phi^{-1}(H)$ is a covering map. The map $g\tilde{H} \to \phi(g)H$ is then a covering map of $\tilde{G}/\tilde{H}$ onto $G/H$. As this map is a local isometry, we see that $G/H$ is locally symmetric. [He, p.213] □

Examples of symmetric Lie algebras

- If $\mathfrak{g}$ is a compact semisimple Lie algebra and $s$ any involutive automorphism of $\mathfrak{g}$ then $(\mathfrak{g}, s)$ is an effective symmetric Lie algebra.

- If $\mathfrak{g}$ is a noncompact semisimple Lie algebra and $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is a Cartan decomposition of $\mathfrak{g}$ (where $\mathfrak{u}$ is the subalgebra) we let $s$ be given by $s(T+X) = T - X$ with respect to this Cartan decomposition. Then $(\mathfrak{g}, s)$ is an effective symmetric Lie algebra.
- If $\mathfrak{e}$ is a finite dimensional vector space of $\mathbf{R}$ and $\mathfrak{u}$ the Lie algebra of a compact Lie subgroup of $GL(\mathfrak{e})$, let $\mathfrak{g}$ be the direct sum $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$. We define a bracket by
$$\begin{array}{ll} [X_1, X_2] = 0 & X_1, X_2 \in \mathfrak{e} \\ [T, X] = -[X, T] = T(X) & T \in \mathfrak{u}, X \in \mathfrak{e} \\ [T_1, T_2] = T_1T_2 - T_2T_1 & T_1, T_2 \in \mathfrak{u} \end{array}$$
Then the mapping $s(T + X) = T - X$ makes $(\mathfrak{g}, s)$ into an effective symmetric Lie algebra (see Helgason, Ch. IV, §5).

## Decomposition of the Symmetric Space

Let $(\mathfrak{g}, s)$ be an effective symmetric Lie algebra and decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ into the $+1$ and $-1$ eigenspaces for $s$. If $\mathfrak{g}$ is compact and semisimple then $(\mathfrak{g}, s)$ is said to be of **compact type**. If $\mathfrak{g}$ is noncompact and semisimple and if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is a Cartan decomposition, then $(\mathfrak{g}, s)$ is said to be of **noncompact type**. If $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$ then $(\mathfrak{g}, s)$ is said to be of **Euclidean type**. If $(G, H)$ is a pair associated to $(\mathfrak{g}, s)$ then we say $(G, H)$ is of the type of $(\mathfrak{g}, s)$. If $M$ is a symmetric space then it is of the type of the pair $(I_0(M), H)$ where $H$ is the isotropy group of some point in $M$. In this section we prove the following decomposition theorem which shows that any effective symmetric Lie algebra can be decomposed into three others. We follow Helgason Chapter V, §1.

**Theorem 3.3.** *Let $(\mathfrak{g}, s)$ be an effective symmetric Lie algebra. Then there exist ideals $\mathfrak{g}_0, \mathfrak{g}_-, \mathfrak{g}_+$ with the following properties:*

1. $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$
2. $\mathfrak{g}_0, \mathfrak{g}_-, \mathfrak{g}_+$ *are invariant under $s$ and orthogonal with respect to the Killing form of $\mathfrak{g}$.*
3. *If $s_0, s_-, s_+$ denote the restrictions of $s$, then the pairs $(\mathfrak{g}_0, s_0)$, $(\mathfrak{g}_-, s_-)$, and $(\mathfrak{g}_+, s_+)$ are effective symmetric Lie algebras of Euclidean type, compact type, and noncompact type, respectively.*

**Corollary 3.4.** *Let $M$ be a simply connected symmetric space. Then $M$ is a product $M_0 \times M_- \times M_+$ where $M_0$ is a Euclidean space, $M_-$ is of compact type, and $M_+$ is of noncompact type.*

*Proof of Corollary.* Let $G = I_0(M)$ and $H$ the isotropy group at a point. Then $M = G/H$. Let $(\tilde{G}, p)$ be the universal cover of $G$ and $\tilde{H}$ the identity component of $p^{-1}(H)$. The map $gH \mapsto p(g)H$ is a covering map $\tilde{G}/\tilde{H} \to G/H$ and since $M$ is simply connected we have $M = \tilde{G}/\tilde{H}$. Let $s = d\sigma$ denote the involutive automorphism of $\mathfrak{g}$, the Lie algebra of $G$. Then $(\mathfrak{g}, s)$ is an effective symmetric Lie algebra. Decomposing $\mathfrak{g}$ as in the theorem we obtain a decomposition $\tilde{G} = G_0 \times G_- \times G_+$. This induces a decomposition $\tilde{H} = H_0 \times H_- \times H_+$ and thus we can define $M_0 = G_0/H_0$, $M_- = G_-/H_-$, and $M_+ = G_+/H_+$. [He, p.244] □

The proof of the theorem follows from a sequence of lemmas, each of which has a simple proof from definitions and small calculations. I will not include proofs here but merely refer the interested reader to Helgason.

Let us first write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ and $\mathfrak{p}$ are the +1 and -1 eigenspaces of $s$. It is immediately obvious that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$, and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$. Let $B$ denote the Killing form of $\mathfrak{g}$. As $B$ is invariant under automorphisms of $\mathfrak{g}$, we see that $\mathfrak{h}$ and $\mathfrak{p}$ are orthogonal with respect to $B$.

**Lemma 3.5.** *The Killing form $B$ is strictly negative definite on $\mathfrak{h}$.*

Now let $K$ denote the subgroup of $Int(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$. Note that $K$ is compact in $GL(\mathfrak{g})$. $K$ is connected and the above bracket relations tell us that $k \cdot \mathfrak{h} \subset \mathfrak{h}$, $k \cdot \mathfrak{p} \subset \mathfrak{p}$ for all $k \in K$. As $K$ is compact, it leaves invariant a strictly positive definite symmetric bilinear form $Q$ on $\mathfrak{p} \times \mathfrak{p}$. Find a basis $X_1, \ldots, X_n$ of $\mathfrak{p}$ so that $Q(X, X) = \sum x_i^2$ for $X = \sum x_i X_i$ and $B(X, X) = \sum \beta_i x_i^2$ for some real numbers $\beta_i$. Now put

$$\begin{aligned} \mathfrak{p}_0 &= \operatorname{span}\{X_i : \beta_i = 0\} \\ \mathfrak{p}_- &= \operatorname{span}\{X_i : \beta_i < 0\} \\ \mathfrak{p}_+ &= \operatorname{span}\{X_i : \beta_i > 0\} \end{aligned}$$

so $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+$ and these subspaces are orthogonal with respect to $Q$ and $B$. Each is invariant under $s$. Defining endomorphism $\mathfrak{B}$ of $\mathfrak{p}$ by $\mathfrak{B}(X_i) = \beta_i X_i$, we see that $Q(\mathfrak{B}X, Y) = B(X, Y)$ on $\mathfrak{p}$. As $B$ and $Q$ are invariant under $K$, $\mathfrak{B}$ commutes with the restriction of each $k \in K$ to $\mathfrak{p}$. Thus $\mathfrak{p}_0, \mathfrak{p}_-, \mathfrak{p}_+$ are invariant under $K$ and $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$.

**Lemma 3.6.** *The subspaces $\mathfrak{p}_0, \mathfrak{p}_-, \mathfrak{p}_+$ satisfy the following properties.*

(1) $\mathfrak{p}_0 = \{X \in \mathfrak{g} : B(X, Y) = 0$ *for all* $Y \in \mathfrak{g}\}$
(2) $[\mathfrak{p}_0, \mathfrak{p}] = \{0\}$ *and* $\mathfrak{p}_0$ *is an abelian ideal in* $\mathfrak{g}$.
(3) $[\mathfrak{p}_-, \mathfrak{p}_+] = \{0\}$.

Now put $\mathfrak{h}_+ = [\mathfrak{p}_+, \mathfrak{p}_+]$, $\mathfrak{h}_- = [\mathfrak{p}_-, \mathfrak{p}_-]$ and let $\mathfrak{h}_0$ be the orthogonal complement (with respect to $B$) of the span of $\mathfrak{h}_+$ and $\mathfrak{h}_-$.

**Lemma 3.7.** *The subspaces $\mathfrak{h}_0, \mathfrak{h}_-, \mathfrak{h}_+$ are ideals in $\mathfrak{h}$, orthogonal with respect to $B$, and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_- \oplus \mathfrak{h}_+$.*

**Lemma 3.8.** *We have*

(1) $[\mathfrak{h}_0, \mathfrak{p}_-] = [\mathfrak{h}_0, \mathfrak{p}_+] = \{0\}$
(2) $[\mathfrak{h}_-, \mathfrak{p}_0] = [\mathfrak{h}_-, \mathfrak{p}_+] = \{0\}$
(3) $[\mathfrak{h}_+, \mathfrak{p}_0] = [\mathfrak{h}_+, \mathfrak{p}_-] = \{0\}$

To prove the theorem, we must consider two cases, $\mathfrak{p}_0 = \{0\}$ and $\mathfrak{p}_0 \neq \{0\}$. Suppose $\mathfrak{p}_0 \neq \{0\}$ . Put

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{h}_0 \oplus \mathfrak{p}_0 \\ \mathfrak{g}_- &= \mathfrak{h}_- \oplus \mathfrak{p}_- \\ \mathfrak{g}_+ &= \mathfrak{h}_+ \oplus \mathfrak{p}_+ \end{aligned}$$

Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$ and the subspaces are invariant under $s$ and orthogonal with respect to $B$. From the lemmas, we have

$$\begin{aligned} [\mathfrak{g}_0, \mathfrak{g}] &= [\mathfrak{h}_0, \mathfrak{g}] + [\mathfrak{p}_0, \mathfrak{g}] \\ &= [\mathfrak{h}_0, \mathfrak{h}] + [\mathfrak{h}_0, \mathfrak{p}_0] + [\mathfrak{h}_0, \mathfrak{p}_+] + [\mathfrak{h}_0, \mathfrak{p}_-] + [\mathfrak{p}_0, \mathfrak{g}] \\ &\subset \mathfrak{h}_0 + \mathfrak{p}_0 + \mathfrak{h}_0 \\ &\subset \mathfrak{g}_0. \end{aligned}$$

We also find that $[\mathfrak{g}_-, \mathfrak{g}] \subset \mathfrak{g}_-$ and $[\mathfrak{g}_+, \mathfrak{g}] \subset \mathfrak{g}_+$ so each of $\mathfrak{g}_0, \mathfrak{g}_-, \mathfrak{g}_+$ are ideals in $\mathfrak{g}$. $B$ is strictly negative on $\mathfrak{g}_-$ so $\mathfrak{g}_-$ is a semisimple compact Lie algebra. $B$ is strictly negative on $\mathfrak{h}_+$ and strictly positive on $\mathfrak{p}_+$ so $\mathfrak{g}_+$ is semisimple and $\mathfrak{g}_+ = \mathfrak{h}_+ \oplus \mathfrak{p}_+$ is a Cartan decomposition. Finally, we examine $\mathfrak{g}_0$. The center $\mathfrak{z}$ of $\mathfrak{g}$ is also the center of $\mathfrak{g}_0$ so $\mathfrak{h}_0 \cap \mathfrak{z} \subset \mathfrak{h} \cap \mathfrak{z} = \{0\}$. We need to see that $\mathfrak{h}_0$ is compactly imbedded in $\mathfrak{g}_0$. We construct a Lie group $G$ with Lie algebra isomorphic to $\mathfrak{g}$. Indeed, if we let $m = \dim \mathfrak{z}$ then $G = Int(\mathfrak{g}) \times \mathbf{R}^m$ has Lie algebra $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g} \times \mathfrak{z}$. Note that $\mathfrak{z} \subset \mathfrak{p}$ and let $\mathfrak{z}^\perp$ be the orthogonal complement (with respect to $Q$) in $\mathfrak{p}$. Then $[\mathfrak{h}, \mathfrak{z}^\perp] \subset \mathfrak{z}^\perp$ and $[\mathfrak{z}^\perp, \mathfrak{z}^\perp] \subset \mathfrak{h}$ so $\mathfrak{h} \oplus \mathfrak{z}^\perp$ is an ideal in $\mathfrak{g}$ isomorphic to $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{g}$. It follows then from 2.15 that $\mathfrak{h}_0$ is compactly imbedded in $\mathfrak{g}_0$. Also $\mathfrak{p}_0$ is an abelian ideal in $\mathfrak{g}_0$ so $(\mathfrak{g}_0, s_0)$ is a symmetric Lie algebra of Euclidean type.

Finally, if $\mathfrak{p}_0 = \{0\}$, $\mathfrak{h}_0$ is an ideal in $\mathfrak{g}$. Its Killing form is strictly negative definite so $\mathfrak{h}_0$ is compact and semisimple. Define

$$\begin{array}{llll} \mathfrak{g}_0 = \{0\} & \mathfrak{g}_- = \mathfrak{h}_0 \oplus \mathfrak{h}_- \oplus \mathfrak{p}_- & \mathfrak{g}_+ = \mathfrak{h}_+ \oplus \mathfrak{p}_+ & \text{if } \mathfrak{p}_- \neq \{0\} \\ \mathfrak{g}_0 = \{0\} & \mathfrak{g}_- = \{0\} & \mathfrak{g}_+ = \mathfrak{h}_0 \oplus \mathfrak{h}_+ \oplus \mathfrak{p}_+ & \text{if } \mathfrak{p}_- = \{0\} \end{array}$$

This completes the proof of the theorem.

## Duality and Sectional Curvature

We point out an interesting duality between symmetric Lie algebras of compact type and noncompact type, following Helgason Ch.V,§2. Let $(\mathfrak{g}, s)$ be a symmetric Lie algebra and decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ into +1 and -1 eigenspaces. Let $\mathfrak{g}^*$ be the subset $\mathfrak{h} + i\mathfrak{p}$ of the complexification $\mathfrak{g}^C$. $\mathfrak{g}^*$ is then a Lie algebra over $\mathbf{R}$. The map $s^* : T + iX \mapsto T - iX$ is an involutive automorphism of $\mathfrak{g}^*$. We say $(\mathfrak{g}^*, s^*)$ is the **dual** of $(\mathfrak{g}, s)$.

**Theorem 3.9.** *Let $(\mathfrak{g}, s)$ be a symmetric Lie algebra. Then*
1. *the pair $(\mathfrak{g}^*, s^*)$ is a symmetric Lie algebra;*
2. *if $(\mathfrak{g}, s)$ is of compact type then $(\mathfrak{g}^*, s^*)$ is of noncompact type and vice versa; and*
3. *if $(\mathfrak{g}_1, s_1)$ is isomorphic to $(\mathfrak{g}_2, s_2)$ then $(\mathfrak{g}_1^*, s_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, s_2^*)$.*

*Proof.* Let $(\mathfrak{g}^C)^R$ be the Lie algebra $\mathfrak{g}^C$ considered as a Lie algebra over $\mathbf{R}$. It has a complex structure $J$ given by multiplication by $i$ on $\mathfrak{g}^C$. As any endomorphism of $\mathfrak{g}$ or $\mathfrak{g}^*$ extends to an endomorphism of $(\mathfrak{g}^C)^R$ commuting with $J$, we may view $GL(\mathfrak{g})$ and $GL(\mathfrak{g}^*)$ as Lie subgroups of $GL((\mathfrak{g}^C)^R)$. The adjoint groups $Int(\mathfrak{g})$ and $Int(\mathfrak{g}^*)$ are also Lie subgroups of $GL((\mathfrak{g}^C)^R)$.

Let $H$ be the subgroup of $Int(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$. $H$ is compact and so $\mathfrak{h}$ is compactly imbedded in $\mathfrak{g}^*$. This gives 1. Note that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are real forms of $\mathfrak{g}^C$ so the Killing forms are obtained by restriction. This gives 2. For 3, note that an isomorphism between $(\mathfrak{g}_1, s_1)$ and $(\mathfrak{g}_2, s_2)$ extends uniquely to an isomorphism of $\mathfrak{g}_1^C$ onto $\mathfrak{g}_2^C$. [He, p.235] □

Now we use the information we have about Lie algebras to understand the geometry of symmetric spaces.

**Theorem 3.10.** *Let $(\mathfrak{g}, s)$ be an effective symmetric Lie algebra and suppose that $(G, H)$ is an associated pair. We assume $H$ is connected and closed. Let $\langle , \rangle$ be a $G$-invariant Riemannian metric on $G/H$.*
1. *If $(G, H)$ is of compact type then $G/H$ has sectional curvature $\geq 0$.*
2. *If $(G, H)$ is of noncompact type then $G/H$ has sectional curvature $\leq 0$.*
3. *If $(G, H)$ is of Euclidean type then $G/H$ has sectional curvature $\equiv 0$.*

*Proof.* As usual, we identify the tangent space to $G/H$ at $p \in \pi(e)$ with $\mathfrak{p}$, the -1 eigenspace for $s$. Let $X, Y$ be orthonormal vectors in $\mathfrak{p}$ with respect to the $G$-invariant structure $\langle , \rangle$. From Proposition 1.7, the sectional curvature is given by

$$K(X,Y) = \langle [[X,Y],X],Y \rangle.$$

The pair $(G,H)$ is of Euclidean type if $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$ so 3 is clear. Assume then that $\mathfrak{g}$ is semisimple. The metric defines a bilinear form $Q$ on $\mathfrak{p}$. As in the proof of Theorem 3.3, let $\mathfrak{B}$ be the endomorphism of $\mathfrak{p}$ so that $Q(\mathfrak{B}X,Y) = B(X,Y)$ for $X, Y \in \mathfrak{p}$. $B$ is as always the Killing form. As $\mathfrak{B}$ is symmetric, its eigenvalues $\beta_i$ are real. Let $\mathfrak{q}_i$ be the eigenspaces of $\mathfrak{B}$. The $\mathfrak{q}_i$ are pairwise orthogonal with respect to $Q$ and $B$.

Let us assume for now that $[\mathfrak{q}_i, \mathfrak{q}_j] = \{0\}$ if $i \neq j$. If this holds, then we can write $[X,Y] = \sum [X_i, Y_i]$, the decomposition with respect to $\mathfrak{p} = \bigoplus \mathfrak{q}_i$ and $[[X_i,Y_i],X] = [[X_i,Y_i],X_i]$. Thus

$$\begin{aligned} K(X,Y) &= \sum \langle [[X_i,Y_i],X_i],Y_i \rangle \\ &= \sum \frac{1}{\beta_i} B([[X_i,Y_i],X_i],Y_i) \\ &= \sum \frac{1}{\beta_i} B([X_i,Y_i],[X_i,Y_i]). \end{aligned}$$

In case 1 we have $\beta_i < 0$ and in case 2 we have $\beta_i > 0$, so the theorem follows.

It remains to see that $[\mathfrak{q}_i, \mathfrak{q}_j] = \{0\}$ if $i \neq j$. Let $\mathfrak{h}$ be the Lie algebra of $H$. $\mathfrak{B}$ commutes element-wise with $\mathrm{ad}_{\mathfrak{g}}\, \mathfrak{h}$ so $[\mathfrak{h}, \mathfrak{q}_i] \subset \mathfrak{q}_i$. Put $X_i \in \mathfrak{q}_i, X_j \in \mathfrak{q}_j$, and $T \in \mathfrak{h}$. Then $[X_i, X_j] \in \mathfrak{h}$ and $B(T, [X_i, X_j]) = B([T, X_i], X_j) = 0$. Since $B$ is strictly negative definite on $\mathfrak{h}$ (Lemma 3.5) we must have $[X_i, X_j] = 0$. [He, p.241] □

## Irreducible Symmetric Lie algebras

Here we make a further decomposition of symmetric Lie algebras of compact or noncompact type. If $(\mathfrak{g}, s)$ is a symmetric Lie algebra, write as before, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ and $\mathfrak{p}$ are the +1 and -1 eigenspaces of $\mathfrak{g}$. The pair $(\mathfrak{g}, s)$ is **irreducible** if

(i) $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ contains no nontrivial ideals of $\mathfrak{g}$.
(ii) the algebra $\mathrm{ad}_{\mathfrak{g}}\, \mathfrak{g}$ acts irreducibly on $\mathfrak{p}$.

An associated pair $(G,H)$ is irreducible if $(\mathfrak{g}, s)$ is irreducible and a symmetric space is irreducible if the pair $(I_0(M), H)$ is irreducible. If $(G,H)$ is an irreducible pair, then any two $G$-invariant Riemannian structures on $G/H$ differ only by a constant factor. Indeed $Ad_G H$ is a compact linear group acting irreducibly on $\mathfrak{p}$. The endomorphism $\mathfrak{B} : \mathfrak{p} \to \mathfrak{p}$ (notation from proof of Theorem 3.3) commutes with each element of $\mathrm{Ad}_G\, H$. Thus $\mathfrak{B}$ can have only one eigenvalue and the forms $Q$ and $B$ are proportional. We will therefore always assume that the Riemannian metric is given by $+B$

or $-B$ where $B$ is the Killing form. We mention an alternate description of criterion (ii).

**Lemma 3.11.** *If (i) is satisfied then (ii) holds if and only if $\mathfrak{h}$ is a maximal proper subalgebra of $\mathfrak{g}$.* [He, p.378]

**Theorem 3.12.** *Let $(\mathfrak{g}, s)$ be a symmetric Lie algebra and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the decomposition into +1 and -1 eigenspaces for $s$. Assume $\mathfrak{g}$ is semisimple and that $\mathfrak{h}$ contains no nontrivial ideals of $\mathfrak{g}$. Then there exist ideals $\mathfrak{g}_i$ of $\mathfrak{g}$ such that*

(1) $\mathfrak{g} = \bigoplus \mathfrak{g}_i$
(2) *The ideals $\mathfrak{g}_i$ are pairwise orthogonal with respect to the Killing form and they are invariant under $s$.*
(3) *If $s_i$ is the restriction of $s$ to $\mathfrak{g}_i$ then each pair $(\mathfrak{g}_i, s_i)$ is an irreducible symmetric Lie algebra.*

*Proof.* The proof is similar to that of Theorem 3.3. Let $Q$ and $\mathfrak{B}$ be as before. Then $\mathfrak{B}$ is symmetric with respect to $Q$, that is, $Q(\mathfrak{B}X, Y) = Q(X, \mathfrak{B}Y)$ for $X, Y \in \mathfrak{p}$. Decompose $\mathfrak{p} = \bigoplus \mathfrak{q}_i$, the eigenspaces of $\mathfrak{B}$. The $\mathfrak{q}_i$ are orthogonal with respect to $Q$ and $B$. Each is invariant under $\mathrm{ad}_{\mathfrak{g}} \mathfrak{h}$ and can be decomposed into irreducible subspaces which are orthogonal with respect to $Q$ and $B$. We thus obtain $\mathfrak{p} = \bigoplus \mathfrak{p}_i$, an orthogonal decomposition where each $\mathfrak{p}_i$ is invariant and irreducible under $\mathrm{ad}_{\mathfrak{g}} \mathfrak{h}$. Put $\mathfrak{h}_i = [\mathfrak{p}_i, \mathfrak{p}_i]$ and $\mathfrak{g}_i = \mathfrak{h}_i \oplus \mathfrak{p}_i$. The rest of the proof is just as in Theorem 3.3. □

**Theorem 3.13.** *The irreducible symmetric Lie algebras of compact type are*

I. *$(\mathfrak{g}, s)$ where $\mathfrak{g}$ is a compact simple Lie algebra and $s$ and involutive automorphism of $\mathfrak{g}$.*
II. *$(\mathfrak{g}, s)$ where compact $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of simple ideals which are interchanged by $s$.*

*The irreducible symmetric Lie algebras of noncompact type are*

III. *$(\mathfrak{g}, s)$ where $\mathfrak{g}$ is a simple noncompact Lie algebra over $\mathbf{R}$, the complexification $\mathfrak{g}^C$ is a simple Lie algebra over $\mathbf{C}$, and $s$ is such that the fixed points form a compactly imbedded subalgebra of $\mathfrak{g}$.*
IV. *$(\mathfrak{g}, s)$ where $\mathfrak{g} = \mathfrak{a}^R$, $\mathfrak{a}$ is a simple Lie algebra over $\mathbf{C}$ and $s$ is the conjugation of $\mathfrak{g}$ with respect to a maximal compactly imbedded subalgebra.*

*Finally, we point out that if $(\mathfrak{g}^*, s^*)$ is the dual of $(\mathfrak{g}, s)$, then*

$$(\mathfrak{g}, s) \text{ is of type III} \iff (\mathfrak{g}^*, s^*) \text{ is of type I}$$
$$(\mathfrak{g}, s) \text{ is of type IV} \iff (\mathfrak{g}^*, s^*) \text{ is of type II}$$

The proof of this theorem is completely straightforward and can be found in [He, p.379].

**Corollary 3.14.** *Let $M$ be a simply connected symmetric space of compact or noncompact type. Then $M$ is a product $M = M_1 \times \cdots \times M_l$ where each $M_i$ is irreducible.*

## 4. Hermitian Symmetric Spaces of Noncompact Type

Let $M$ be a connected manifold with almost complex structure $J$. A Riemannian metric $\langle , \rangle$ on $M$ is said to be **Hermitian** if $\langle JX, JY \rangle = \langle X, Y \rangle$ for vector fields $X$ and $Y$, and it is **Kahlerian** if also $\nabla_X J = 0$. Hermitian means that $J$ acts by isometries and Kahlerian means that $J$ is invariant under parallelism. It should be noted that if $M$ is a complex manifold, its complex structure gives rise to an almost complex structure $J$. If we define a 2-form $\omega(X, Y) = \langle X, JY \rangle$ on $M$ then the Riemannian structure is Kahlerian if and only if $\omega$ is closed. [He, p.358].

Now let $M$ be a connected complex manifold with a Hermitian structure. $M$ is said to be a **Hermitian symmetric space** if for each $p \in M$ there exists a holomorphic isometry $S$ fixing $p$ and such that $dS_p = -I$ on $T_pM$, the real tangent space. We denote by $A(M)$ the group of holomorphic isometries. As with the real symmetric space, the identity component $G = A_0(M)$ acts transitively on $M$. For $p \in M$ and $S$ the symmetry about $p$, let $H$ be the isotropy subgroup of $p$ in $G$. The automorphism $\sigma(g) = SgS$ makes the pair $(G, H)$ symmetric and $M = G/H$.

**Lemma 4.1.** *The Hermitian structure of a Hermitian symmetric space is Kahlerian.* [He, p.372]

Let $G$ be a connected Lie group and $H$ a closed subgroup of $G$. Suppose $J$ is a $G$-invariant almost complex structure on $M = G/H$ and put $p = \pi(e)$. $J_p$ is a linear transformation on $T_pM$ such that

(i) $J_p^2 = -I$ and
(ii) $J_p$ commutes with the elements of the linear isotropy group $H^*$.

Now suppose $J_p$ is an endomorphism of $T_pM$ satisfying (i) and (ii). $M = G/H$ has then a unique $G$-invariant almost complex structure. In fact, this structure is integrable, as the following proposition shows:

**Proposition 4.2.** *Let $(G, H)$ be a symmetric pair and $p = \pi(e) \in G/H = M$. Let $\langle , \rangle$ be any $G$-invariant Riemannian metric on $M$. Suppose $J_p$ is an endomorphism of $T_pM$ such that*

(1) $J_p^2 = -I$
(2) $\langle J_pX, J_pY \rangle = \langle X, Y \rangle$ *for all* $X, Y \in T_pM$
(3) $J_p$ *commutes with the elements of the linear isotropy group* $H^*$.

*Then $J_p$ extends to a unique $G$-invariant almost complex structure $J$ on $M$. The Riemannian structure is Hermitian and $J$ is integrable. Therefore, as a complex manifold, $M$ is a Hermitian symmetric space.*

*Proof.* Existence and uniqueness of the almost complex structure is clear. The inner product is Hermitian by property 2 since both it and $J$ are $G$-invariant. Let $\sigma$ be any involutive automorphism of $G$ such that $F_0^\sigma \subset H \subset F^\sigma$. As in the proof of Theorem 1.5 the symmetry $S$ about $p$ is defined by $S \circ \pi = \pi \circ \sigma$. To see that $S$ preserves the almost complex structure, let $q \in M$ and $v \in T_qM$. In the notation of Theorem 1.5, find $g$ so that $\tau_g p = q$

and set $v_p = d\tau_{g^{-1}}(v)$. Then the invariance of $J$ under $G$ and the relations $S \circ \tau_g = \tau_{\sigma g} \circ S$ and $dS_p \circ J_p = J_p \circ dS_p$ show that

$$\begin{aligned}
dS_q(J_q v) &= dS_q(J_q d\tau_g v_p) \\
&= dS_q d\tau_{\sigma g}(J_p v_p) \\
&= d\tau_{\sigma g} J_p(dS_p v_p) \\
&= J_{S(q)} d\tau_{\sigma g} dS_p(v_p) \\
&= J_{S(q)} dS_q d\tau_g(v_p) \\
&= J_{S(q)}(dS_q v).
\end{aligned}$$

To see that $J$ is integrable, we must show that

$$[X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] = 0 \qquad (*)$$

for vector fields $X$ and $Y$ on $M$. As $M$ is homogeneous, we need only check this at the point $p \in M$. Let $U$ be a normal neighborhood of $p$ and for vector $X_p \in T_pM$, extend to a vector field on $U$ so that $X_q$ is the parallel translate of $X_p$ along the unique geodesic joining $p$ and $q$. If $Y$ is a vector field on $U$ also obtained by parallel translation, then

$$[X, Y]_p = (\nabla_X Y)_p - (\nabla_Y X)_p = 0.$$

It suffices to check $(*)$ for vector fields of this type since the left-hand side is $C^\infty(M)$-linear. If $X$ and $Y$ are formed by parallel translation then so are $JX$ and $JY$ by the $G$-invariance of $J$. Relation $(*)$ follows. $J$ is therefore a complex structure on $M$ and the symmetry about any point $q$ in $M$ given by $S^q = gSg^{-1}$ (where $g \in A(M)$ is such that $g(p) = q$) is holomorphic. Therefore, $M$ is a Hermitian symmetric space. [He, p.373] □

Let $M$ be a Hermitian symmetric space. $M$ is said to be of **compact type** or **non-compact type** depending on the type of the symmetric pair $(A_0(M), H)$, $H$ the isotropy group of a point in $M$.

**Theorem 4.3.** *Let $M$ be a simply connected Hermitian symmetric space. Then $M$ is a product $M = M_0 \times M_- \times M_+$ where each factor is a simply connected Hermitian symmetric space. $M_0 = \mathbf{C}^m$ for some $m$, $M_-$ is of compact type, and $M_+$ is of non-compact type.*

*Proof.* Putting $G = A_0(M)$ the proof begins exactly as in the proof of Corollary 3.4. We write $M = \tilde{G}/\tilde{H}$ and if $\mathfrak{g}$ is the Lie algebra of $G$, we decompose $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+$. We obtain decompositions $\tilde{G} = G_0 \times G_- \times G_+$ and $\tilde{H} = H_0 \times H_- \times H_+$ which lead to our decomposition of $M$. Let $\mathfrak{p}, \mathfrak{p}_0, \mathfrak{p}_-, \mathfrak{p}_+$ denote the -1 eigenspaces for $s$ which can be identified with the tangent spaces of $M, M_0, M_-, M_+$, respectively. We have $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+$. Let $J$ be the almost complex structure on $M$. The inner product $\langle , \rangle$ on $M$ gives a bilinear form $Q$ on $\mathfrak{p} \times \mathfrak{p}$ and $J_p$ is an endomorphism of $\mathfrak{p}$. For $Y \in \mathfrak{p}_0$, write $JY = Y_0 + Y_- + Y_+ \in \mathfrak{p}_0 \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+$. Let Ad denote the adjoint representation of $\tilde{G}$. We have $\mathrm{Ad}(h)Y = Y$ for any $h$ in $H_- \times H_+$ by Lemma

3.8. Since $\mathrm{Ad}(h)$ and $J_p$ commute we find that $\mathrm{Ad}(h)(Y_- + Y_+) = Y_- + Y_+$ for any $h$ in $H_- \times H_+$. This in fact holds for any $h$ in $H$ since if $h \in H_0$, $\mathrm{Ad}(h)$ fixes all of $\mathfrak{p}_- \oplus \mathfrak{p}_+$. As a consequence we find that $Y_- + Y_+ = 0$; indeed, it follows from the proof of Theorem 3.3 that if $X \in \mathfrak{p}$ commutes with all elements in $\mathfrak{h}$ we must have $X \in \mathfrak{p}_0$. Therefore $J_p(\mathfrak{p}_0) \subset \mathfrak{p}_0$. Also, $J_p$ leaves $Q$ invariant so $J_p(\mathfrak{p}_- \oplus \mathfrak{p}_+) \subset \mathfrak{p}_- \oplus \mathfrak{p}_+$. Arguing similarly we see that $\mathfrak{p}_-$ and $\mathfrak{p}_+$ are also invariant under $J_p$. Finally, Proposition 4.2 implies that $M_-$ and $M_+$ are Hermitian symmetric. [He, p.374] □

As in Section 3 we can decompose even further, into irreducible components.

**Theorem 4.4.** *Let $M$ be a simply connected Hermitian symmetric space of compact type or non-compact type. Then $M$ is a product $M = M_1 \times \cdots \times M_l$ where the factors are simply connected Hermitian and irreducible.*

*Proof.* We first state two Lemmas whose proofs can be found in Helgason.

**Lemma 4.5.** *If $M$ is a Hermitian symmetric space, then $I_0(M)$ is semisimple if and only if $A_0(M)$ is semisimple. In this case, $A_0(M) = I_0(M)$.* [He, p.374]

**Lemma 4.6.** *If $H$ denotes the isotropy group in $A_0(M)$ of a point $p$ in $M$ and $H^*$ the corresponding linear isotropy group, then the complex structure $J_p$ of $T_pM$ belongs to the center of the Lie algebra of $H^*$.* [He, p.376]

To prove the theorem, Corollary 3.14 allows that we need only show the factors are Hermitian. Using the decomposition of the Lie algebra $\mathfrak{g}$ of $A_0(M) = I_0(M) = G$ to give a decomposition of $\tilde{G}$ and $\tilde{H}$, we can decompose our almost complex structure $J = J_1 \times \cdots \times J_l$. Each $J_i$ lies in the center of the Lie algebra of $H_i$. Each $H_i$ is connected so the group $\mathrm{Ad}_{G_i} H_i$ commutes element-wise with $J_i$. Proposition 4.2 thus implies that the $M_i$ are Hermitian. [He, p.381] □

### Symmetric spaces of non-compact type

Here we show that for a noncompact simple Lie algebra $\mathfrak{g}$ over $\mathbf{R}$, there exists a unique symmetric space $M$ of noncompact type such that $I_0(M)$ has Lie algebra $\mathfrak{g}$. This $M$ is diffeomorphic to a Euclidean space; in particular, it is simply connected. It turns out that all Hermitian symmetric spaces are simply connected, regardless of type. This is a consequence of the following theorem for the noncompact case; the compact case is discussed in [He, p.376].

Let $\mathfrak{g}$ be a noncompact semisimple Lie algebra over $\mathbf{R}$ and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a Cartan decomposition. Defining $s : T + X \mapsto T - X$ with respect to this decomposition, the pair $(\mathfrak{g}, s)$ becomes a symmetric Lie algebra of noncompact type. Recall that an associated pair $(G, H)$ is symmetric if $H$ is closed, $\mathrm{Ad}_G H$ is compact, and if there exists an involutive automorphism $\sigma$ of $G$ with $F_0^\sigma \subset H \subset F^\sigma$. Such a $\sigma$ is unique (Remark B of Section 1).

**Theorem 4.7.** *Suppose $(G, H)$ is any pair associated to $(\mathfrak{g}, s)$ as above. Then*

(1) *$H$ is connected and closed.*
(2) *There exists an involutive automorphism $\sigma$ of $G$ with fixed point set $H$ and such that $d\sigma = s$. The pair $(G, H)$ is symmetric.*
(3) *The map $\phi : \mathfrak{p} \times H \to G$ given by $(X, h) \mapsto (\exp X)h$ is a diffeomorphism and the exponential map $\operatorname{Exp}$ on $M$ is a diffeomorphism of $\mathfrak{p} = T_pM$ onto the symmetric space $M = G/H$.*

**Corollary 4.8.** *Let $M, M'$ be symmetric spaces of noncompact type such that the groups $I_0(M), I_0(M')$ have the same Lie algebra $\mathfrak{g}$. With Riemannian structures given by the Killing form of $\mathfrak{g}$, $M$ and $M'$ are isometric.*

*Proof of Corollary.* $M$ and $M'$ are given by two Cartan decompositions of $\mathfrak{g}$. If these decompositions are the same, we simply apply the theorem. If not, the two Cartan decompositions are conjugate by (2.19) via an inner autormorphism of $\psi$ of $\mathfrak{g}$. It is possible to define the isometry $M \to M'$ with $\psi$. □

*Proof of Theorem.* Let $H_0$ be the identity component of $H$. As in Theorem 3.2, $H_0$ is closed (see [He, p.213]) and $G/H_0$ is a locally symmetric space. The exponential maps satisfy $\pi(\exp X) = \operatorname{Exp} X$ where $X$ is in $\mathfrak{p}$ and $\pi$ is the projection $G \to G/H_0$. This shows completeness of $G/H_0$ and so $\operatorname{Exp} : \mathfrak{p} \to G/H_0$ is surjective; thus $\phi : \mathfrak{p} \times H_0 \to G/H_0$ is surjective. We aim to show that $\phi$ is injective on $\mathfrak{p} \times H$.

Let $X_1, X_2 \in \mathfrak{p}$ and $h_1, h_2 \in H$ be so that $(\exp X_1)h_1 = (\exp X_2)h_2$. Applying Ad we see that $e^{\operatorname{ad} X_1} \circ \operatorname{Ad}(h_1) = e^{\operatorname{ad} X_2} \circ \operatorname{Ad}(h_2)$. With respect to the positive definite symmetric bilinear form $B_s(Y, Z) = -B(Y, sZ)$ for $Y, Z \in \mathfrak{g}$, we find that $e^{\operatorname{ad} X_i}$ is symmetric and positive definite while $\operatorname{Ad}(h_i)$ is orthogonal. Such a decomposition is unique so $\operatorname{ad} X_1 = \operatorname{ad} X_2$. The center of $\mathfrak{g}$ is trivial so we obtain $X_1 = X_2$. We find also that $h_1 = h_2$ so $\phi$ is injective on $\mathfrak{p} \times H$. We have $\phi(\mathfrak{p} \times H_0) = \phi(\mathfrak{p} \times H)$ so injectivity implies that $H = H_0$; that is, $H$ is connected.

We remark that $H$ contains the (discrete) center $Z$ of $G$ and $H$ is compact if and only if $Z$ is finite. These statements are proved in [He, p.253].

Let $\tilde{G}$ be the universal cover of $G$ and let $\mathcal{S}$ be the unique automorphism of $\tilde{G}$ with $d\mathcal{S} = s$. The map $\mathcal{S}$ induces a map $\sigma$ on $G$ which makes $(G, H)$ into a symmetric pair. Finally, to prove 3 it suffices to show that the bijective $\phi$ is everywhere regular. This follows from a computation which can be found in [He, p.254]. □

### Bounded symmetric domains

A bounded domain $D \subset \mathbf{C}^n$ is **symmetric** if each point $p$ in $D$ is an isolated fixed point of a holomorphic involution of $D$. Here we achieve our main goal: the Hermitian symmetric spaces of noncompact type are exactly the bounded symmetric domains in $\mathbf{C}^n$. As a consequence of the proof of this theorem, we find that any noncompact Hermitian symmetric space $M$ can be imbedded as an open subset of its compact "dual" $M^*$. As an example, we may keep in mind the Poincare disk $\{|z| < 1\}$ sitting inside the Riemann sphere $\hat{\mathbf{C}}$. We follow the outline of [IT, p.247] and refer often to the details in [He, Ch. VIII §7]. Unless otherwise specified, the page number references in the proof are from Helgason.

**Theorem 4.9.** (i) *Each bounded symmetric domain $D$, equipped with the Bergman metric, is a Hermitian symmetric space of noncompact type. In particular, every bounded symmetric domain is simply connected.*
(ii) *Let $M$ be a Hermitian symmetric space of noncompact type. Then $M$ is biholomorphic to a bounded symmetric domain in $\mathbf{C}^n$.*

*Proof (i).* For any domains $\Omega$ in $\mathbf{C}^n$, we can define a Hermitian structure known as the Bergman metric for which biholomorphic mappings $\Omega_1 \to \Omega_2$ are isomotries. See [K, §1.4] for a clear expository on the Bergman kernel and metric; details can also be found in [pp.364-371]. If $D$ is a bounded symmetric domain then its holomorphic automorphisms are isometries of the Bergman metric; thus $D$ is a Hermitian symmetric space. To see that $D$ is of noncompact type, let $\tilde{D}$ be the universal cover. As in Corollary 3.4, decompose $\tilde{D} = M_0 \times M_- \times M_+$. $M_0$ is a Euclidean space and the restriction of the covering map to $M_0 \times \{x\} \times \{y\}$ is a bounded holomorphic map into $\mathbf{C}^n$ and thus constant. Therefore $M_0$ is a point. Similarly we can show that $M_-$ is degenerate since Hermitian symmetric spaces of compact type are compact complex manifolds [IT, p.258]. $D$ is therefore of noncompact type and so $D = \tilde{D}$ is simply connected.

*Proof (ii).* Let $M$ be a Hermitian space of noncompact type, $(G, H)$ the corresponding symmetric pair where $G = A_0(M) = I_0(M)$, and $(\mathfrak{g}, s)$ its effective symmetric Lie algebra over $\mathbf{R}$. We decompose into eigenspaces of $s$, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Let $\mathfrak{z}$ be the center of $\mathfrak{h}$ and $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{h}$. Then $\mathfrak{z} \subset \mathfrak{t}$ and $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$; indeed, the centralizer of $\mathfrak{z}$ in $\mathfrak{g}$ contains $\mathfrak{h}$ but is not all of $\mathfrak{g}$ and so must be exactly $\mathfrak{h}$ by Lemma 3.11 [p.383]. We will exploit the relationship between the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and the root space decomposition of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$. Let $\mathfrak{t}^C$,$\mathfrak{h}^C$, and $\mathfrak{p}^C$ denote the corresponding complexifications of the subalgebras.

Put $\mathfrak{g}^* = \mathfrak{h} + i\mathfrak{p}$, a compact real form of $\mathfrak{g}^C$ and let $\tau$ denote conjugation with respect to $\mathfrak{g}^*$. The Hermitian form

$$B_\tau(X, Y) = -B(X, \tau Y)$$

is strictly positive definite on $\mathfrak{g}^C$ and

$$B_\tau([Z, X], Y) = B_\tau(X, [Z, Y])$$

for any $Z$ in $\mathfrak{g}^*$. We find that endomorphisms $\operatorname{ad} T$ on $\mathfrak{g}^C$ are semisimple for each $T$ in $\mathfrak{t} \cup i\mathfrak{t}$. Since they all commute, $\mathfrak{t}^C$ is a Cartan subalgebra of $\mathfrak{g}^C$. Let $\Delta$ be the non-zero roots of $\mathfrak{g}^C$ with respect to $\mathfrak{t}^C$. For $\alpha \in \Delta$, we have $(\mathfrak{g}^C)^\alpha \subset \mathfrak{h}^C$ or $(\mathfrak{g}^C)^\alpha \subset \mathfrak{p}^C$ and thus the decompositions [p.384]

$$\mathfrak{h}^C = \mathfrak{t}^C \oplus \bigoplus_{(\mathfrak{g}^C)^\alpha \subset \mathfrak{h}^C} (\mathfrak{g}^C)^\alpha$$

$$\mathfrak{p}^C = \bigoplus_{(\mathfrak{g}^C)^\beta \subset \mathfrak{p}^C} (\mathfrak{g}^C)^\beta$$

Introduce compatible orderings on the duals of real vector spaces $i\mathfrak{h}$ and $i\mathfrak{z}$. Let $\Delta^+$ be the set of positive roots and $\Delta_\mathfrak{z}$ the roots which do not vanish identically on $\mathfrak{z}$. Define

$$\mathfrak{p}_+ = \bigoplus_{\beta \in \Delta^+ \cap \Delta_\mathfrak{z}} (\mathfrak{g}^C)^\beta$$

$$\mathfrak{p}_- = \bigoplus_{-\beta \in \Delta^+ \cap \Delta_\mathfrak{z}} (\mathfrak{g}^C)^\beta$$

**Lemma 4.10.** *$\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}^C$ such that* [p.384]

$$[\mathfrak{h}^C, \mathfrak{p}_-] \subset \mathfrak{p}_-, \quad [\mathfrak{h}^C, \mathfrak{p}_+] \subset \mathfrak{p}_+, \quad \mathfrak{p}^C = \mathfrak{p}_- \oplus \mathfrak{p}_+.$$

Alternately, $\mathfrak{p}_-$ and $\mathfrak{p}_+$ can be described as follows. There exists a unique $Z \in \mathfrak{h}$ so that the complex structure on $\mathfrak{p}$ is given by $J = \operatorname{ad}_\mathfrak{p} Z$ and $\mathfrak{h} = \{X \in \mathfrak{g} : [Z, X] = 0\}$. Then $\mathfrak{p}_\pm = \{X \in \mathfrak{g}^C : [Z, X] = \mp iX\}$ [IT, p.243].

We may now write $\mathfrak{g}^C = \mathfrak{h}^C \oplus \mathfrak{p}_- \oplus \mathfrak{p}_+$. Let $G^C$ be the connected complex Lie subgroup of $GL(\mathfrak{g}^C)$ generated by $\operatorname{ad} \mathfrak{g}^C$. Since the center of $G$ is trivial and the Lie algebra of $G^C$ is identified with $\mathfrak{g}^C$ via the adjoint representation, we may identify $G = A_0(M)$ with the Lie subgroup of $G^C$ generated by $\mathfrak{g}$. Let subgroups $G^*, H^C, P_-, P_+$ correspond to Lie algebras $\mathfrak{g}^*, \mathfrak{h}^C, \mathfrak{p}_-, \mathfrak{p}_+$ respectively.

**Lemma 4.11.** *The exponential map is a diffeomorphism of $\mathfrak{p}_-$ onto $P_-$ and of $\mathfrak{p}_+$ onto $P_+$* [p.388].

**Lemma 4.12.** *The map $(q, h, p) \mapsto qhp$ is a diffeomorphism of $P_- \times H^C \times P_+$ onto an open submanifold of $G^C$ containing $G$* [p.388].

**Lemma 4.13.** *$H^C P_+$ is closed in $G^C$, the set $GH^C P_+$ is open in $P_- H^C P_+$, and $G \cap H^C P_+ = H$* [p.390].

**Lemma 4.14.** *The map $gH \mapsto gH^C P_+$ is a holomorphic diffeomorphism $G^*/H \to G^C/H^C P_+$. The manifold $M^* = G^*/H$ is a compact Hermitian symmetric space with symmetric Lie algebra dual to that of $M$* [p.393].

From the lemmas we obtain the well-defined holomorphic **Borel imbedding**

$$\phi : M = G/H \hookrightarrow G^C/H^C P_+ = M^*$$

where the image $\phi(M)$ is an open subset. As $\phi$ is $G$-equivariant, any element of $G = A_0(M)$ extends uniquely to a holomorphic transformation of $M^*$. Now consider the composition mapping $\pi \circ \exp : \mathfrak{p}_- \to M^*$ given by $(\exp X)H^C P_+$. This is a holomorphic imbedding such that $\phi(M) \subset \pi \exp(\mathfrak{p}_-)$. We can therefore define $\psi : M \to \mathfrak{p}_-$ uniquely by $\phi(p) = \pi \exp(\psi(p))$. The map $\psi$ is known as the **Harish-Chandra imbedding**.

We need only to demonstrate that $\psi(M)$ is a bounded domain in the complex vector space $\mathfrak{p}_-$. For $X \in \mathfrak{p}$ denote by $|X|$ the operator norm for $\operatorname{ad} X$ on $\mathfrak{g}$ with respect to the inner product $B_\tau$ defined above. Define the real vector space isomorphism $j : \mathfrak{p} \to \mathfrak{p}_-$ by $X \mapsto 1/2(X - iJX)$ and a norm on $\mathfrak{p}_-$ by $\|j(X)\| = |X|/2$. Finally, it can be shown that

$$\psi(M) = \{X \in \mathfrak{p}_- : \|X\| < 1\}$$

[IT, p.249]. □

To make more sense of the algebra in the proof of this theorem, we conclude this section with an example. Let $D$ be the open unit disk in $\mathbf{C}$. We follow [IT, pp.182,236,250] and [He, p.394].

$$\begin{aligned} G &= SU(1,1) &= \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\} \\ H &= SO(2) &= \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : |a|^2 = 1 \right\} \end{aligned}$$

where $\sigma(A) = (A^{-1})^t$ on $G$. Note that $D = G/H$ as a smooth manifold. The Lie algebra of $G$ is

$$\begin{aligned} \mathfrak{g} &= \mathfrak{su}(1,1) \\ &= \left\{ B \in M_2(\mathbf{C}) : \operatorname{Tr} B = 0 \text{ and } \bar{B}^t \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} B \right\} \\ &= \left\{ \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix} : c = -\bar{c} \right\}. \end{aligned}$$

In determining the decomposition of $\mathfrak{g}$ we find

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} : z \in \mathbf{C} \right\}$$

and the complex structure is as

$$J \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & iz \\ -i\bar{z} & 0 \end{pmatrix}.$$

Complexifying, we obtain $\mathfrak{g}^C = \mathfrak{sl}(2, \mathbf{C})$ and

$$\mathfrak{p}_- = \left\{ \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} : z \in \mathbf{C} \right\}.$$

If we identify $\mathfrak{p}_-$ with $\mathbf{C}$, the Harish-Chandra imbedding is the identification of $G/H$ with the open unit disk. Let us continue with the computations to determine the role of the "dual" $D^*$ which will be the Riemann sphere $\hat{\mathbf{C}}$.

$$G^* = SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

and we also have $G^C = SL(2,\mathbf{C})$ and $H^C = SO(2,\mathbf{C})$. $G^C$ acts on the Riemann sphere by Mobius transformations and

$$H^C P_+ = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : ad = 1 \text{ and } c \in \mathbf{C} \right\}$$

is the isotropy group of the origin. We have a diffeomorphism $D^* = G^*/H \hookrightarrow G^C/H^C P_+ = \hat{\mathbf{C}}$ given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} H \mapsto \frac{b}{\bar{a}}$$

as $b/\bar{a}$ is the image of the origin under the linear fractional transformation $\frac{az+b}{-\bar{b}z+\bar{a}}$. Let us note that the $G$-orbit of the origin is the unit disk, which describes the Borel imbedding into the Riemann sphere

$$G/H \hookrightarrow G^C/H^C P_+.$$

Finally $P_-$ is identified with $\mathbf{C}$ inside $\hat{\mathbf{C}}$.

## 5. The Classification

As mentioned in the Introduction, a complete classification of the symmetric spaces is possible. The decomposition theorems of Section 3 allow us to restrict attention to the irreducible symmetric spaces. As in [He, Ch.X§1], the classification of simply connected, irreducible symmetric spaces up to isometry is equivalent to the classification of irreducible symmetric Lie algebras up to isomorphism. We assume the Riemannian metric is induced by the Killing form.

We have seen tht all symmetric spaces of noncompact type are simply connected and described uniquely by a symmetric Lie algebra of noncompact type. By the duality, it suffices to classify the symmetric spaces of compact type. We say an irreducible symmetric Lie algebra is of type I, II, III, or IV as described by Theorem 3.13. Helgason shows, for example, that the symmetric spaces of type II are exactly the compact connected simple Lie groups with bi-invariant metric [He, p.439].

If $M$ is a symmetric space and $\tilde{M}$ its universal cover, then $M = \tilde{M}/U$ where $U$ is a discrete subgroup of the center of $G$. The centers can be described by their root systems [He, Ch.X §3]. The problem of classification is thus reduced to the following problems:

(1) Find all compact simple Lie algebras $\mathfrak{g}$, up to isomorphism.
(2) Find all involutive automorphisms of $\mathfrak{g}$, up to conjugation in $Aut(\mathfrak{g})$.
(3) Find all centers of compact, simple, simply connected Lie groups.

Since any semisimple Lie algebra over $\mathbf{C}$ has a compact real form, unique up to an inner automorphism, problem 1 is equivalent to:

1a. Find all simple Lie algebras over $\mathbf{C}$, up to isomorphism.

These are listed at the end of Section 2. If $\mathfrak{u}$ varies in the set of compact real forms of semisimple $\mathfrak{g}$ over $\mathbf{C}$ and $s$ runs through the possible involutive automorphism of $\mathfrak{u}$, the dual $(\mathfrak{u}^*, s^*)$ to $(\mathfrak{u}, s)$ runs through all noncompact real forms of $\mathfrak{g}$. Problem 2 is then equivalent to both of the following:

2a. For each simple Lie algebra over $\mathbf{C}$, find all noncompact real forms, up to isomorphism.
2b. For each simple Lie algebra over $\mathbf{C}$, find all involutive automorphisms, up to conjugation in $Aut(\mathfrak{g})$.

[He, Ch.X §1]. For a complete list of symmetric spaces see [B, p.200]. In his notation, the Lie subalgebra $\mathfrak{k}$ is our $\mathfrak{h}$. See also Helgason or Loos [L].

## References

[B] Besse, Arthur L. *Einstein Manifolds.* Springer-Verlag, 1978.
[He] Helgason, Sigurdur. *Differential Geometry, Lie groups, and Symmetric Spaces.* Academic Press, 1978.
[Hu] Humphreys, James E. *Introduction to Lie algebras and Representation Theory.* Springer-Verlag, 1972.
[IT] Ise, Mikio and Masaru Takeuchi. *Lie Groups I & II.* American Mathematical Society, 1991.
[K] Krantz, Steven. *Function Theory of Several Complex Variables.* Wadsworth and Brooks/Cole, 1992.
[L] Loos, Ottmar. *Symmetric Spaces I & II* W.A. Benjamin, Inc., 1969.
[O] O'Neill, Barrett. *Semi-Riemannian Geometry.* Academic Press, 1983.
[P] Petersen, Peter. *Riemannian Geometry.* Springer, 1998.
[S] Serre, Jean-Pierre. *Complex Semisimple Lie algebras.* Springer-Verlag, 1987.