# Chapter 2: The Concept of Length

## Units

In dealing with scientific problems, we must often know such things as the location of an object, the distance between objects, how tall or wide an object is, or how fast it is moving. Central to all these ideas is the concept of length. In this tutor, we shall examine the various disguises in which length can appear.

First, however, we must note the messy problem of units. The kinds of units chosen for length were for a long time quite arbitrary. The cubit, used by several ancient civilizations, was the length of a forearm between the tip of the middle finger and the elbow. The fathom was the width of a Viking sailor's embrace. (Fathom that!)

The foot was a convenient length defined as the length of a person's foot. Of course, everyone has a different-sized foot, so if we want to use, say, the king's foot as a standard, we will have to mark it permanently; we cannot very well lug the king around! Since the king is the ruler of the land, it was both rational and proper to call this marker a ruler as well. To show students the chaos caused by not having a standard length, have them measure the length of their desks in cubits or the width of the room in feet with each child using his or her own body measurements. You will have as many different values as measurers.

*The first known standard foot measure was from Sumeria, where a definition is given in a statue of Gudea of Lagash from around 2575 BC.*

## Non-Standard Units

Many mathematics curricula introduce children to measurement by using non-standard units. For example, you could have children measure the length of a classroom using their own foot as a unit of measure. A child's foot is very non-standard, since there will be feet of many different sizes in the class. There are several children's books and a Math Trailblazer Adventure Book story that show what problems one can run into when measuring with different feet. The moral is that we need a standard foot and in ancient times a standards was chosen but using the foot of some king. (So we can truly say that we measure length with a "ruler.")

*The average foot length is about 240 mm (9.4 in) for current Europeans.*

Children can also be introduced to measurement of length using plastic links.

A link is shown in Figure 1 along with a chain of links. The measurement of the object shown is 4 1/2 links. The reasons we use links are: (1) to make counting the unit length easy and (2) to keep the numbers manageable. And counting links is easy since they are so large (about 1 1/2 inches long).

By alternating link colors (2 reds, 2 whites, 2 reds, etc.), one can make the counting even easier by skip counting by 2s. If you want to have the children skip count by 5s, then have a chain with 5 blues, 5 yellows, 5 blues, etc.

With links, it is also easy to measure to the nearest half-link as shown in Figure 1. If the edge we are measuring is near the middle of the link, then the length is 4 1/2 links. If shorter, then L = 4 links; if longer, L = 5 links.

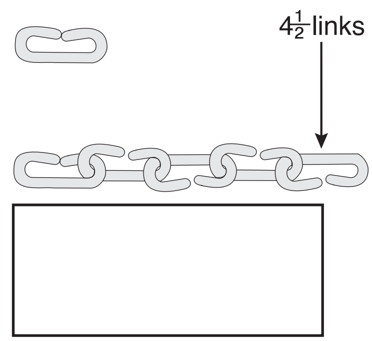



Figure 1: Links

## The Metric System

*In 1983, the 17th General Conference on Weights and Measures redefined the meter to be the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second.*

Of course, links are not a commonly used standard of measure. In the United States, a common unit of measure is the **foot**. Unfortunately, many objects are smaller than a foot. This means one must subdivide the standard unit. For some miserable reason, the smaller unit, the **inch**, is 1/12 of a foot. Multiplying and dividing by 12 is a pain in the neck, so it is tedious to convert inches to feet and feet to inches. Also, the answer can be ugly. For example, 20 inches equals 1.66666666…. feet. (OK, so it's 1 1/3, but that's still difficult to work with.) So, the French scientific community at the time of the French Revolution (c. 1790) chose as the standard unit of length a distance which they called a **meter**. It was chosen so that $10^7$ (10 million) metersticks laid end to end would just fit between the North Pole and the equator, as shown in Figure 2. A platinum-iridium rod was constructed with two marks a meter apart and stored in a vault near Paris. Every meterstick, albeit indirectly, comes from this standard. Having defined the meter, the French were smart enough to define all subsequent subdivisions of the meter as integral powers of ten. The inch is divided into halves, quarters, eighths, and other equally horrible numbers; for the meter, the divisions are tenths, hundredths, and thousandths. We shall now explore this point further.

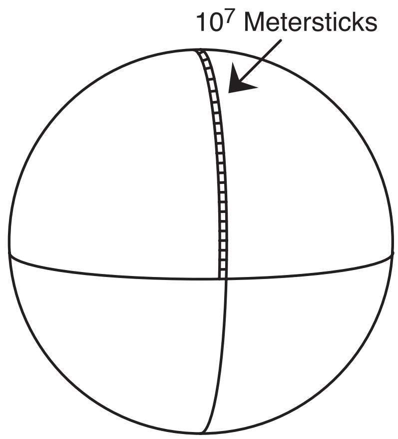



Figure 2: Metersticks from the North Pole to the equator

For measuring ordinary objects in the classroom, the meter is divided into three other units; the **millimeter** (mm), the **centimeter** (cm), and the **decimeter** (dm, although the latter is rarely used). Without being specific as to the size of each unit, we can order them using the greater than (>) or less than (<) sign. Starting with the smallest, we have:

1 millimeter < 1 centimeter < 1 decimeter < 1 meter.

Reversing the order and starting with the largest, we have:

1 meter > 1 decimeter > 1 centimeter > 1 millimeter.

It is important that the child know at least this much before going on to more exact relations.

The key to the subdivision of the meter is the prefix *milli*. *Milli* is related to the word mile. Mile was the distance it took a Roman soldier to step off 1000 paces, a pace being two steps. Since an average pace is approximately 5 feet (try it and see), a mile would be approximately 5000 feet. The crucial point is the number 1000 as related to the word mile. *Milli* is the prefix for "1/1000 of." Thus, a millimeter is one-thousandth of a meter. (Is a millipede one tenth of a centipede?) The word mile has a similar history. The original definition of "mile" was the distance covered in 1000 paces (by a Roman soldier, I seem to recall). So 1000 paces was called a "mille." Over time, this became "mile." There are 1000 millimeters in a meter just as there are 1000 paces in a mile. In each case, we have a subunit that is one-thousandth of the main unit. The word mile is to remind us that there are 1000 paces in a mile. The word *millimeter* is to remind us that there are 1000 millimeters in

*The mile that we know and use today goes back to Queen Elizabeth I of England, who redefined the mile from 5000 feet to 8 furlongs (5280 feet) by statute in 1593.*

a meter. One should try to picture this in one's mind. Millimeters are tiny; it takes a lot of them to make up any macroscopic length. (By macroscopic, we are referring to something one can see unaided versus microscopic where one would need a magnifying glass or microscope to see it.) On the other hand, because a meter is large, a millimeter is usually a fraction of most lengths one would measure in the lab. Thus, it is reasonable that a third-grader knows how to measure the length of his thumb as 3 cm or 30 mm, but not as 0.03 m.

*To measure lengthy distances, the **kilometer** (1000 meters) is a more frequently used unit.*

How are mm, cm, dm, and m related? Let's start with the smallest and see how many mm are in a cm, a dm, and a meter.

1 centimeter contains 10 mm;

1 decimeter contains 100 mm;

1 meter contains 1000 mm.

Based on these relationships, we should be able to figure out how many centimeters are in a decimeter or in a meter. Here, however, the French have made it easy for us; the prefix for each word gives the answer away. *Centi* stands for 100th and *deci* stands for one-tenth. Thus one *centimeter* is one-hundredth of a meter; there are 100 cm in a meter. A *decimeter* is one-tenth of a meter; there are 10 dm in a meter. What this boils down to then is the following:

1 decimeter contains 10 cm;

1 meter contains 100 cm;

1 meter contains 10 dm.

Everything depends upon the size of the meter. Once that is fixed (by our rod in Paris), the sizes of all other metric units are determined.

## Measuring Length

The simplest way to get started in the metric system is to count, using a meterstick, the number of mm or cm in a given length. For example, suppose we measure the width of this sheet of paper in cm. Then a careful student with a good meter stick will see that the width of the page is between 21 and 22 cm. If you stick to whole centimeters, then for students who are not yet comfortable with fractions or decimals, this is all you can say. Young students are often comfortable measuring to the nearest half centimeter. As students begin to learn decimals, they should be able to determine that the width of this page "is" 21.6 cm. The reason for the quotation marks around "is" is that in reality the measurment is only 21.6 cm, to the nearest tenth of a centimeter. You would need a much better ruler, with the centimeter divided into hundredths to get more precise measurements. This is a basic feature of all real world (as opposed to word problem) measurements. They are approximations, limited by the precision of the measuring device.

If you want to avoid decimals, you can use millimeters instead of centimeters and get 216 mm. In fact, one of the neat things about the metric system is that you can often choose units so that your measurement is expressed without resort to fractions or decimals. On the other hand, you can purposely choose units that will give decimal or fractional answers. As we just saw, in cm units, the width of the page is a decimal, 21.6 cm. We could have asked for the width in meters. Since the width is less than a meter, we are dealing with fractions. In this case, the width is 0.216 meters or roughly 1/5 of a meter. Clearly, there is great potential in the metric system for teaching math and linking this to scientific measurement.

With regard to addition, when adding numbers they must always have the same units. For example:

(a) 5 cm + 6 m = ? This is a "no-no"; the units are mixed.

(b) 5 cm + 600 cm = 605 cm.
This is okay—we are adding the same units.

(c) 8 mm + 50 cm = ?
We should convert the cm to mm and get:

(d) 8 mm + 500 mm = 508 mm.

Thus, if we ask a student to measure the length of his arm by separately measuring his hand (say in mm), his lower arm (say in cm), his upper arm (say in decimeters), and then adding them, he will first have to convert to a set of consistent units that he or she can handle. If you do not choose to have the students work with fractions, the students can change the units to millimeters as shown above. If you want to give the students a chance to work on decimal fractions, they can change the units to centimeters.

.8 + 50 cm = 50.8 cm

This brings us to another point—how to use a ruler. At first it seems quite apparent: just place the end of the ruler at the end of the object and read the length directly (Figure 3). However, a better test of whether students really understand how to use a ruler as well as a test of their ability to subtract is to place the object in the center of the ruler. Clearly, the length of the object should not depend upon its position vis-a-vis the ruler, but we have found that many young people (and even a few at our university) have trouble understanding how to find the length in the latter case.

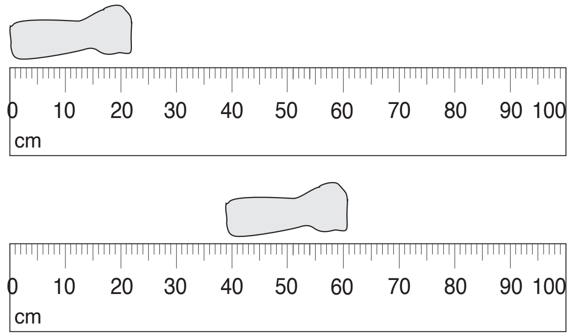
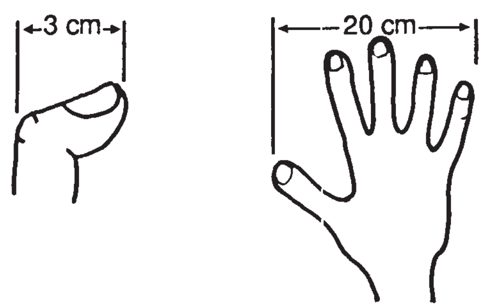



Figure 3: How to use a ruler

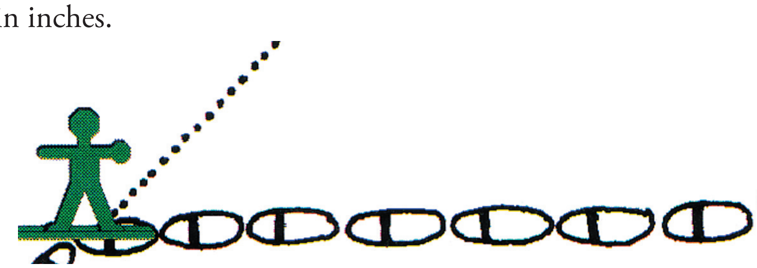
## Estimating Length

Since we do not always have a ruler handy, a few "natural" rulers might be fun to discuss and use. For example, say the length of a person's upper thumb from knuckle to tip is generally about 3 cm while his spread-out fingers span about 20 cm, as shown in Figure 4. Either can now be used to measure the length of an object. Of course, there is the foot, a convenient measure for stepping off distances.

Figure 4: Natural rulers




According to Wikipedia, the foot was first defined in the 12th century by King Henry I, who wished to standardize the unit of measurement in England. Not surprisingly, it was defined to be the length of *his* foot. If you know your feet are 9 inches long, you can estimate a distance by stepping it off, heel-to-toe, counting the number of steps and multiplying by 9 to get the distance in inches.



## Chapter 2 Questions

1) Some examples of commonly used words that are associated with measurement and length are *distance* and *height*. Can you think of other words that involve the measurement of length?

2) Find five examples of units of length that have not been previously mentioned. Define them in terms of units we already know (if possible) and describe what these units of length are used for.

3) At Wrigley Field, the distance from home plate to the left field foul pole is 355 feet. At the now defunct Olympic Stadium in Montreal, Canada, the distance from home plate to the left field foul pole is 99.0 meters (the sign on the left field wall really does indicate the length in meters). In which ballpark is it harder to hit a home run down the left field line? (And don't say Olympic Stadium just because the Expos no longer play baseball there.)

4) You overhear an argument between two first graders. "I'm three and a half feet tall," the first child says. "Well, I'm forty-two inches tall, so I'm taller than you are because forty-two is is a bigger number than three and a half," the second one replies. You must intervene immediately. How can you convince your first grade students that they are both the exact same height?

5) The word distance is often used as a synonym for length. That is, someone might say that measuring the distance from Chicago to New York City is equivalent to measuring the length of the straight line that connects the two cities. Why is there ambiguity with such a statement? Provide some suggestions for how this statement can be made less ambiguous.

6) Suppose you wanted to measure the length (or perimeter) of several curved objects, such as a cylinder, a bowling pin, and a bottle of juice. However, you only have a straight ruler at your disposal. For each of these objects, what could you do to measure the length (perimeter) as closely as possible?