Chapter 3: Averages

Introduction

"Average" is one of those words that mean different things to different people. Baseball players talk about their batting averages. A teacher might confide to a colleague that "Jim is just an average student." At the university, students always want to know what the class average is on an exam. Sometimes you hear the statement that the temperature will be about average for this time of year. When someone asks you how you feel, you may reply, "Just average." In everyday usage, "average" is a word that can be anything from a synonym for "typical," "normal," or "usual," to a number derived according to some formula or rule. This TIMS Tutor attempts to lay out some of the different numerical meanings of "average" and to explain some of the importance of averages in mathematics and science.

The Mean – A Wage Dispute

In his wonderful little book, *How to Lie with Statistics*, Darrell Huff gives the example of a factory owner and his workers who are arguing over wages. There are 25 workers including the owner. The owner pays himself $45,000. The others make $15,000, $10,000, $10,000, $5700, $5000, $5000, $5000, $3700, $3700, $3700, $3700, $3000, $2000, $2000, $2000, $2000, $2000, $2000, $2000, $2000, $2000, $2000, $2000, and $2000. The owner says the average wage is $5700 but the workers claim the average wage is only $3000. Even though prices have gone way up since Huff wrote his book 40 years ago, the discrepancy is clear: there is a big disagreement over what people are being paid, let alone what they should be paid.

So, what's going on here? Who's right, the owner or the workers? Both! The factory owner's average is the (arithmetic) mean; the workers' average is the median. There are other averages too, such as the mode (the number or value that occurs most often in a data set). However, in Math Trailblazers, we primarily use the mean and/or the median when finding average values. These two averages will be the focus of our discussion in this tutor.

Usually when people use the term "average," they are referring to the mean. This is the familiar add-up-all-the-numbers-and-divide average you learned in school. The mean has many useful properties that make it beloved by schoolteachers, statisticians, and scientists alike.

Consider finding the average height of all the children in a class. Data for 23 children from a third-grade class is shown in Figure 1.

Name	H Height (in cm)
Karina	133
Federico	136
Ramon	135
Kiela	127
Aesha	126
Bravlia	128
Zuzia	133
Anthony	139
Iorta	146
Cordeli	135
Mary	137
Gennice	124

Name	H Height (in cm)
Curtis	139
Brian C	125
David	135
Anna	141
Brian M	131
Adriana	137
Boberto	134
Lucas	131
Gennifer	129
Amber	129
Nathan	134

Figure 1:
Data from a third-grade class

The mean height of these 23 children is

$$\langle H \rangle = \frac{\text{sum of heights}}{\text{number of children}}$$

$$= \frac{(133 + 136 + ... + 129 + 134)\text{ cm}}{23}$$

$$= \frac{3064\text{ cm}}{23}$$

$$= 133.2\text{ cm}$$

Although no child may have this mean height, it gives everyone a point of departure for making comparisons. For scientists, the mean is often the first number they calculate when looking at a data sample.

There is another way to interpret the mean height of this class. This may seem strange, but consider the following situation. Suppose you walk into another class and find 23 students all to be exactly the same height. The mean is clearly that height. If the mean height of this new class and the mean in our class are the same, then the sum of the heights of the children is the same. That is, imagine making two stacks of the children, one for our class and one for the new class. Stack the children one on top of another. Then the two stacks would be the same height. So, if two class means are the same and the number of children in each class is the same, then the sums of the heights in the classes are the same, even if the individual heights that make the sums are vastly different.

Statisticians like the mean because it is often the "best" estimate of an unknown quantity like a length, an area, or a mass. For example, suppose you are trying to measure the mass of a large steel sphere. You have an unbiased balance, and being well versed in TIMS you know that you need to take repeated readings to get an accurate measurement. So, suppose you make eleven measurements and find the sphere's mass to be 129 gm, 133 gm, 132 gm, 130 gm, 128 gm, 129 gm, 130 gm, 131 gm, 130 gm, 129 gm, and 131 gm. Then

$$\frac{(129 + 133 + \ldots + 130 + 129 + 131)\text{ gm}}{11}$$

$$= \frac{1432\text{ gm}}{11}$$

$$= 130\text{ gm}$$

is the best estimate of the "true" mass of the sphere, given these measurements. (Notice that the quotient above is actually equal to 130.18181818… gm. However, since our original measurements are to the nearest gram, it makes sense to give the average only to the nearest gram.)

Similar situations arise all the time in everyday life. Our factory owner is using the mean; he computes the average wage by adding up everyone's salary (including his big fat one) and dividing by the total number of workers:

$$\frac{\$142{,}500}{25} = \$5{,}700$$

As one final example, suppose Marty scores 85, 84, 86, 87, 84, 87, 85, 85, 83, and 85 on 10 rounds of golf and Ellen scores 83, 83, 95, 97, 81, 83, 82, 84, 96, and 84. Then Marty's mean score is

$$= \frac{85 + 84 + 86 + 87 + 84 + 87 + 85 + 85 + 83 + 85}{10}$$

$$= 85.1$$

and Ellen's mean score is

$$= \frac{83 + 83 + 95 + 97 + 81 + 83 + 82 + 84 + 96 + 84}{10}$$

$$= 86.8$$

So, it would appear that Marty is a better golfer than Ellen.

But anyone who plays golf and bets will notice that if Marty and Ellen played 10 rounds against each other and scored as above, then Marty would win three times and Ellen would win seven times. And the workers in the factory still feel underpaid, despite that nice mean salary. So, to get some perspective on what's happening here, let's look at another average, the median.

The Median

The median is the number in the middle. That is, roughly speaking, the median splits a set of numbers into two halves: one half is less than the median; the other half is more than the median. So, to find the median, rank the numbers from smallest to largest and take the one in the middle.

Suppose, for example, you want to find the median selling price of homes in your town. You go to the town office and find that 15 homes have been sold in the last six months. The selling prices, in rank order, were $67,000, $78,000, $82,000, $85,000, $92,000, $97,000, $112,000, $118,000, $125,000, $132,000, $133,000, $139,000, $167,000, $175,000, and $186,000. The median price was $118,000: seven houses sold for less than $118,000 and seven houses sold for more than $118,000.

When there is an even number of values, then the median can be a little harder to find. Say only 14 of the homes above to have been sold. Suppose that the highest-priced house, the one costing $186,000, went unsold. What is the new median? Counting up seven from the $67,000 house, we end up at the $112,000. Counting down seven from the $175,000 house, we end up at $118,000. The 7th house from the bottom is not the 7th house from the top! What to do? Easy. Go halfway between the two numbers closest to the middle. Halfway between $112,000 and $118,000. That comes out to be $115,000. So the median price of the 14 homes that sold would be $115,000.

Going back to our factory workers' example, there were 25 employees including the owner. The thirteenth salary (from the top or from the bottom) is $3000: there are 12 employees who make less than $3000, and 12 who make more than $3000. So, the median salary is $3000.

Sometimes the median is a better average to use than the mean. One common situation where the median may be preferred is when there are extreme values. In such cases, the mean can give a distorted picture of the "average" because the extreme values tend to "pull" the mean away from where the typical values are. Statisticians say the median is more "robust" than the mean; that is, the median is less affected by extreme values than the mean. This is why the factory workers prefer the median: it gives a truer picture of what a typical worker earns than the mean, which is pulled up by the owner's big salary.

The median offers some insight into our golfing example, too. Marty's (ranked) golf scores were 83, 84, 84, 85, 85, 85, 85, 86, 87, and 87. The two numbers in the middle are 85 and 85, so Marty's median score is 85. Ellen's scores were 81, 82, 83, 83, 83, 84, 84, 95, 96, and 97. The two numbers in the middle are 83 and 84, so Ellen's median score is 83.5. Thus, using medians, Ellen is a slightly better golfer, on average. Every few rounds she has a high score, but her typical score is better than Marty's typical score, so she usually wins.

Obviously, the median has several advantages with respect to the mean: it is less affected by extreme values; it requires little computation; it can be a better indicator of what is typical. You will want to be flexible, sometimes using the mean, sometimes using the median.

Averaging in the Math Trailblazers Classroom

So how are averages used in Math Trailblazers? Suppose you have second-grade students who collected the data shown in Figure 2 for the TIMS Laboratory Experiment, Rolling Along in Centimeters. There are three experimental values for each car, and the students need some measure of the middle that fairly represents their data for each car. In this case, the median is a perfectly respectable average.

T Type of Car	D Distance (in cm/unit)			
	Trial 1	Trial 2	Trial 3	Median
Red	83	93	86	
Blue	44	44	43	
Orange	39	53	46	
Black	194	199	189	

Figure 2: Data from Rolling Along in Centimeters

One way to get a median from data like this is illustrated in Figure 3. The students cross out the largest and smallest numbers in each row and use the one that is left as their average value. Finding the median value of the three trials does not require young children to do any arithmetic and conceptually illustrates quite nicely the idea of an average representing a middle value.

T Type of Car	D Distance (in cm/unit)			
	Trial 1	Trial 2	Trial 3	Median
Red	~~83~~	~~93~~	(86)	86
Blue	~~44~~	(44)	~~43~~	44
Orange	~~39~~	~~53~~	(46)	46
Black	(194)	~~199~~	~~189~~	194

Figure 3: Finding the median

Finding the median can be illustrated more concretely by using links (or string, adding machine tape, etc.) to create "lengths" that match the actual distance that the cars roll each time. At the end of three rolls, the students will have three lengths of links that each represent the distance a car rolled in a trial.

This is illustrated in Figure 4. The median value can then be found by comparing the lengths and discarding the shortest and longest; the remaining length represents the median. A process somewhat like this is used in the first-grade experiment *Rolling Along with Links*.

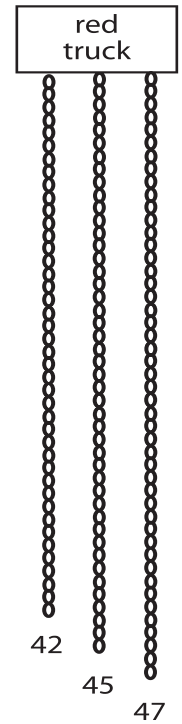
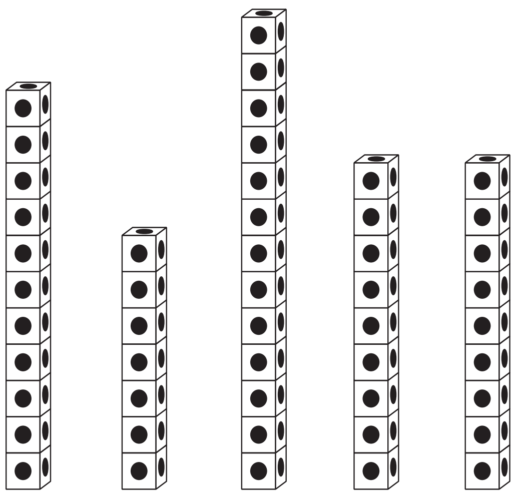


Figure 4

In fourth grade, students begin finding the mean as well as the median. Again the concepts are introduced concretely. For example, students find the median value for several towers of cubes by lining them up from shortest to tallest and selecting the tower in the middle. This is illustrated in Figures 5a and 5b. Figure 6 illustrates the case when there is an even number of trials. To find the median here, students have to find the halfway point between the two middle towers.



Figures 5a and 5b: Finding the median with cubes

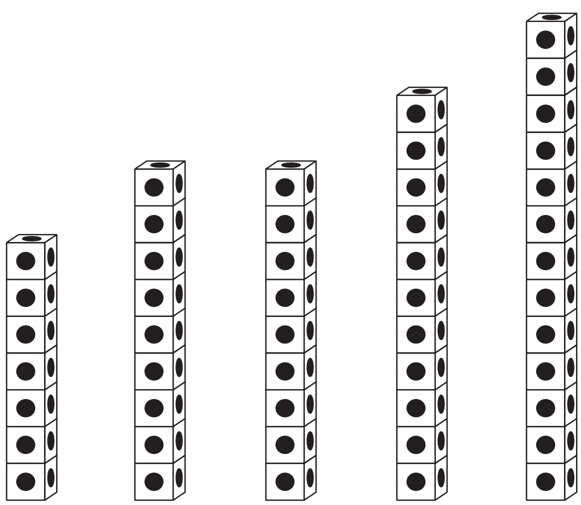


Figure 6: The median number of cubes is 7 1/2.

Cubes are also used when introducing the mean. For example, suppose students begin with the same five towers that are illustrated in Figure 5a. They first estimate the mean by figuring out a way to "even out the towers," as illustrated in Figure 7. Then they calculate the mean using the numerical values. In this manner, the concept of the mean as a representative measure is stressed rather than focusing purely on the arithmetic procedure.

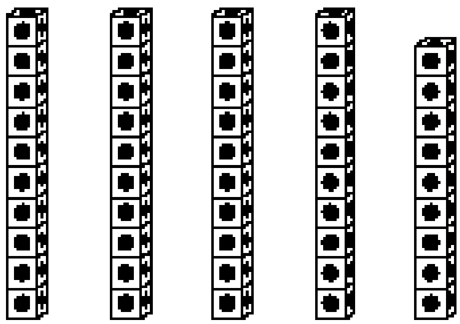


Figure 7: Estimating the mean by "evening out" towers of cubes

Students will often use calculators to calculate the mean of a data set. For example, the mean distance for the red car in Figure 2, when found by calculator, is:

$$\frac{(83 + 93 + 86)\text{ cm}}{3} = 87.333333\text{ cm}$$

This is an example of too much of a good thing. In this case, the calculator gives many more digits than are meaningful. The students have to determine an appropriate level of significance. Keep in mind that our goal is to get a reasonable number to use as an average. Here, the original measurements were made to the nearest cm so it makes sense to also report the mean value to the nearest cm.

Why Average?

Finding the average is a critical step in a Math Trailblazers experiment. It is an important step from raw data to understanding. Errors are inevitable when measurements are made, and averaging is one way to detect and control error. If a child makes the same measurement three times and gets 67 cm, 64 cm, and 84 cm, she should notice that the 84 cm is way off. She can detect her error and redo that measurement.

Even when all the measurements are close, the average minimizes error.

If the measurements are 67 cm, 64 cm, and 67 cm, then the mean, 66 cm, is a more accurate prediction of future measurements than any of the actual measurements.

Another function of averaging is to aggregate raw data. Unlike computers, people cannot handle too many numbers at once. If I tell you the income of every child in a school, then you will be overwhelmed by numbers. If I tell you the median income of children in the school, you know less but probably understand more. The situation in a typical Math Trailblazers experiment is similar. From 12 actual measurements in Rolling Along in Centimeters, we cut down to three averages which are plotted. Then we may even go further, to a single number, the slope of the best-fit line. But that's another story.