Chapter 4: The Concept of Area

Defining Area

The area of a shape or object can be defined in everyday words as the "amount of stuff" needed to **cover** the shape. Common uses of the concept of area are finding the amount of tile needed to cover a floor, the amount of wallpaper needed to cover a wall, and the amount of paint needed to cover a ceiling. Areas of different objects can often be compared directly, without measurement. For example, if one piece of carpet completely covers another, we know the top piece has more area.

Just as with length, in order to measure area we need a unit of measure. Many different units have been used throughout history. The acre is still used as a measure of land area, along with square miles. The square inch, square foot, and square yard are area units in the English system. For example, square yards is the unit of area measure for carpeting in the United States. Most of the world and all scientists use the metric system. For the classroom, the most frequent metric unit of area is the square centimeter. It is defined as the amount of flat surface within a square that is 1 cm on a side. This is illustrated in Figure 1.

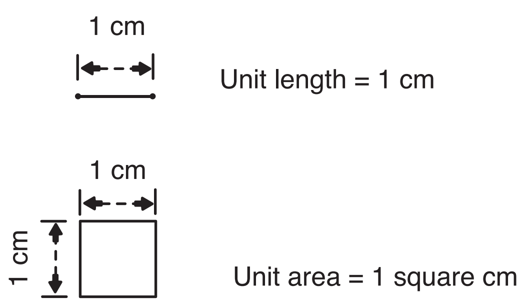
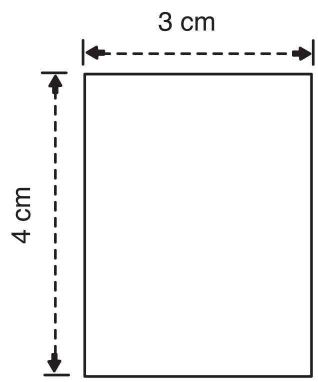


Figure 1: Units of length and area

A note on terminology: Scientists often use the term "centimeters squared" in place of "square centimeters." This may be due to the fact that one way of writing the symbol for square centimeters is cm^2. We prefer to use *sq cm* for square centimeters. On occasion, the use of cm^2 can confuse students. Since 7 squared is 49, they may reason that 7 centimeters squared is 49 square centimeters. Unfortunately, the symbols 7 cm^2 and $(7cm)^2$ sound the same when spoken, if you say cm^2 as "centimeters squared." For that reason, we stick to "square centimeters" and sq cm.

Measuring Area by Counting Square Centimeters - Part 1



One way to find an area is to count the number of sq cm needed to cover a surface. To count the number of sq cm, you can construct unit squares within the desired area. For example, the rectangle in Figure 2 is 3 cm wide and 4 cm high. You can find its area by completing the following steps.

Figure 2: A rectangle

(1) Draw grid of lines so the each square in the grid is 1 cm on a side. (Figure 3.)

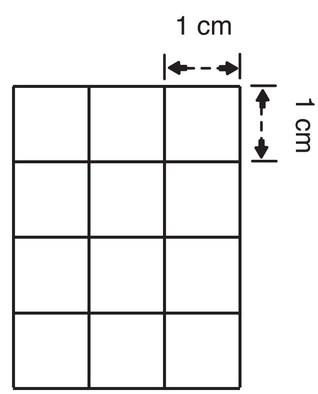


Figure 3: A rectangle tiled with centimeter squares

(2) Count the number of sq cm enclosed in the rectangle. (Figure 4.)

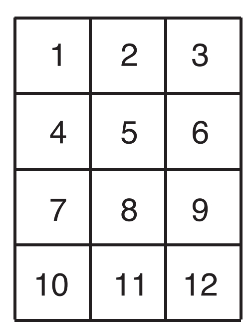


Figure 4: Counting square centimeters

There are 12 sq cm in a rectangle that is 3 cm wide and 4 cm high. So the area of the rectangle is 12 sq cm.

Why Not Use Length x Width?

You may be wondering why we did not use the formula length X width to find the area of the rectangle. After all, 4 X 3=12. The formula works. Indeed, the reason the formula works is precisely because the rectangle can be represented by an ordered array of squares, 3 squares in each row and 4 rows of 3 squares each.

In any ordered array, you can count the total number of elements by multiplying the number of rows by the number of columns. Figure 5 shows 24 apples arrayed in 6 rows. Instead of counting each apple, we take advantage of the array and multiply 4 X 6 to obtain 24 apples. You may use this trick often. For instance, when you buy stamps and want to check to see if the number you purchased was correct, you may multiply the rows by the columns. Using the length times width formula does give the area of a rectangle, but we delay teaching the formula for two reasons. First, we want students to build a mental image of the concept of area. Premature use of the formula for area of rectangles leads to rote use of the formula without understanding. In particular, many students are led to believe that the definition of area is length times width and that this formula works for any shape. While there are formulas for the areas of rectangles, circles, and other geometric shapes, there is no formula for the area of a leaf!

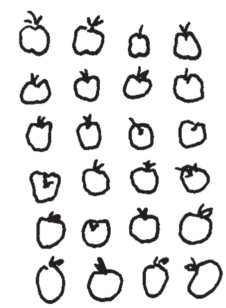


Figure 5: An array of apples

Furthermore, using the formula can be harder than counting square centimeters when the sides of the rectangle are not whole numbers.

For example, what is the area of the rectangle, 3 1/2 cm wide and 2 1/2 cm high, shown in Figure 6? We asked a group of sixth-graders to find the area of this rectangle. Even thought the students knew the length X width formula, 80% could not find the area. They were not able to multiply fractions. It is likely that these students would have been able to find the area by counting sq cm, as we shall see in the next section.

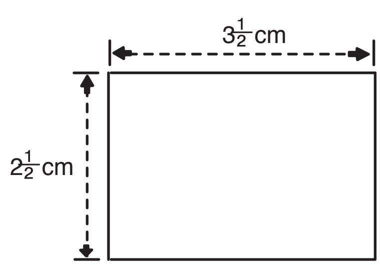


Figure 6: Another rectangle

Counting Square Centimeters - Part 2

In Figure 7, we have set up the square cm grid. To keep track of the sq cm, we first number the whole square centimeters. There are six whole sq cm in the rectangle. Next, we turn to the fractions of sq cm. Students can manipulate sq cm pieces of paper to complete the task. The two half sq cm on the right make up the seventh sq cm and so both are number 7. The two half sq cm along the bottom make up the eighth sq cm and are numbers 8. One half sq cm and one fourth square cm make up the remaining areas. The result is A=8 3/4 sq cm.

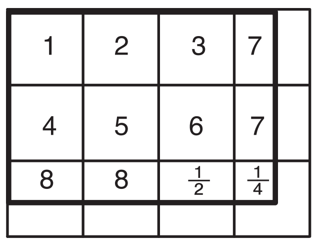


Figure 7: Whole and part centimeter squares

Most students can count squares to find area-no multiplication is necessary. But what if there is no “order” to the figure? The shape is Figure 8 has no unique length or width; the formula length X width does not apply. The only way to find the area is to count the number of sq cm contained within the boundary of the figure.

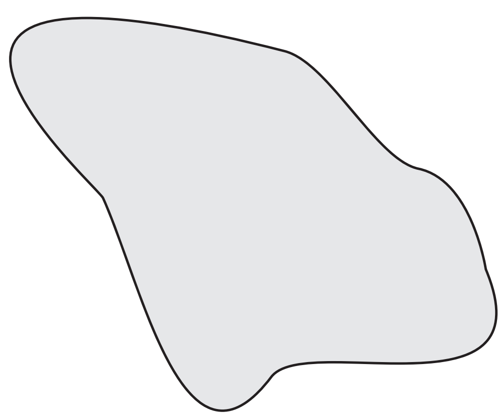


Figure 8: A blob

First, construct a sq cm grid to fit over the shape of Figure 8. It's often easier if the horizontal and vertical boundary of the grid each touch the figure at one point. The grid should extend beyond the figure. The completed grid would look similar to the grid in Figure 9.

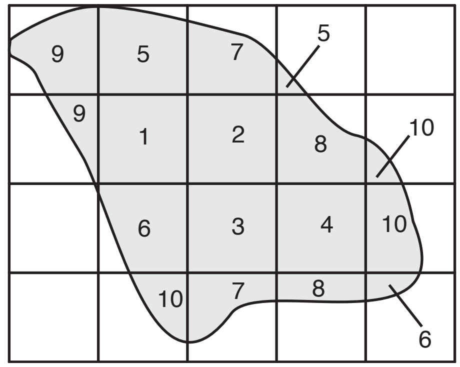


Figure 9: Finding area by counting centimeter squares

Next, count and number the whole sq cm. There are four. Now estimate which fractions of grid squares add together to form one square centimeter. For example, square centimeters number 5 is a big piece on the left and a smaller piece in the lower right corner. Square centimeter number 7 is two pieces, each around a half square cm. Square centimeter 8 is made up of a 3/4 and 1/4 square cm piece. So is square 9. Three pieces make up square centimeters 10, two half square cm pieces and a smaller piece to the right.

Now, all of the shape is covered and counted in square centimeters. The shape has an area of about 10 square cm. While the method does not always give an exact area, the result is usually close. And, the primary benefit of this method is that students will have the opportunity to "see" area, aiding their understanding of this important mathematical concept.

Surface Area

Often, students who understand area quite well seem to have difficulty with the notion of surface area. One problem may be that they have been led to believe that area and surface area are two different things. This is not surprising, since we use two different words. However, area and surface area are identical; a measure of the number of sq cm (of square units) needed to cover an object. Customarily, the term "surface area" is usually used for three dimensional shapes and "area" for two-dimensional shapes.

To measure the surface area of a 3-D object, we count the number of square centimeters needed to cover it, just as with flat shapes. In some cases, this is easy as in the case of a rectangular box (since it is made up of flat pieces). Another easy example can be made by taking a sheet of flat paper and rolling it to make a cylinder (Figure 10). As long as the edges do not overlap, the

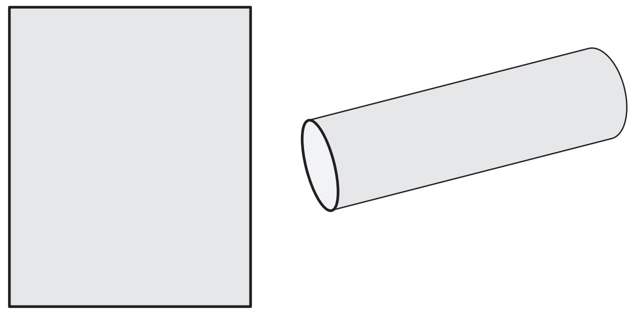


Figure 10: Paper flat and rolled

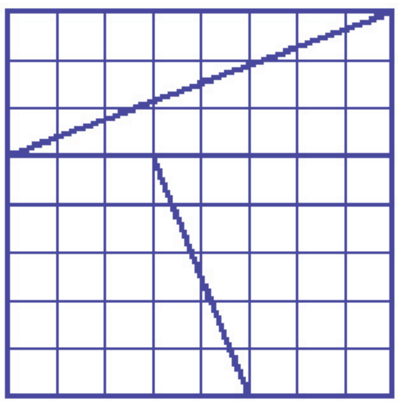
surface area of the outside of the cylinder will be the same as the flat piece of paper. Another way to find the area of a cylinder is to cover it with one square centimeter "stamps". As with flat shapes, you may need some fractional pieces. With more complex shapes, like a sphere, it is hard to get an exact measurement of the surface area, but we can approximate the surface area by covering the object with square centimeters or smaller squares. Another approach, for example to find the surface area of an orange, is to peel the object and "flatten" the "skin" on a square unit grid.

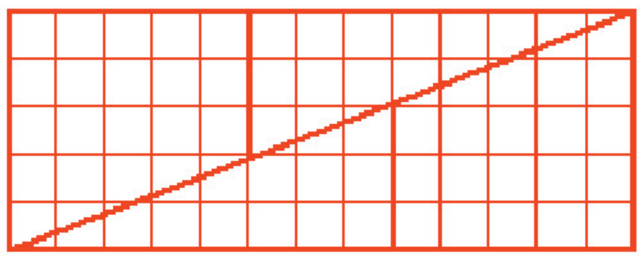
Figure 11: Finding the surface area of an orange by peeling it and flattening out.

Chapter 4 Questions

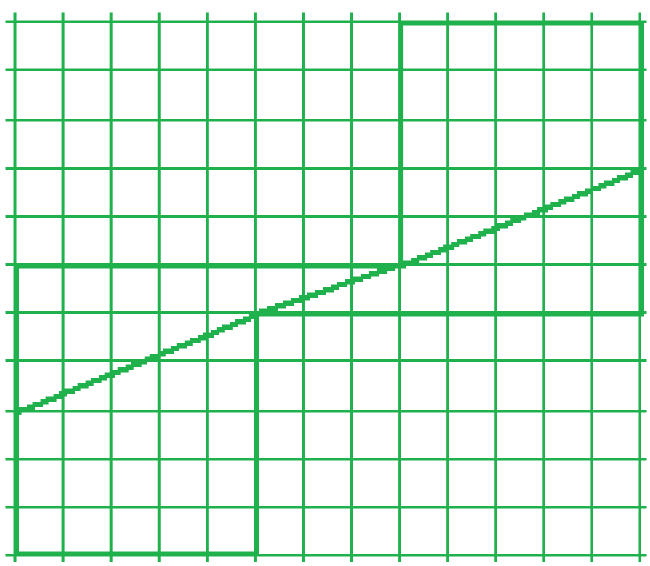
For questions 1 and 2, you will need to consider about the mathematical idea of *dissection*; that is, any two rectilinear figures with equal area can be dissected into a finite number of pieces to form each other. In layman's terms, you can cut up one plane figure into smaller pieces, rearrange those pieces, and get another plane figure with exactly the same area as the first figure.

1) The 8-by-8 square in the diagram here can be cut up into 4 pieces that, when rearranged, make the 5-by-13 rectangle. But the square contains 8 x 8=64 little squares whereas the rectangle contains 5 x 13=65. Where has the extra square come from?





2) The jigsaw above of area 64 little squares, when re-arranged into the rectangle with 65 little squares, had seemingly gained a square. Here is another arrangement. This time the square puzzle's pieces have been re-arranged as shown here and now it loses a square -- there are two 5-by-6 rectangles and 3 squares in a row joining them, making a total area of 63! So what's happened this time???



3) When you did the *Circumference vs. Diameter* lab, you were able to find the length of a curved object's circumference by measuring the circumference with string, "straightening out" the string, and using a ruler. When measuring area, however, the notion of "flattening out" is more complicated. To measure the surface area of a sphere or a spherical object, what method do you think will produce the most accurate results?

4) (Challenge) As you already know, the surface of the Earth is curved. This is the challenge cartographers face when they are trying to draw maps - how can one "flatten out" the surface of the Earth with as little distortion as possible? Distortion creates two problems: the area of land masses is distorted or directions (north, south, east, west) are not preserved. Find examples of maps that a) have as little area distortion as possible and b) preserve directions as best as possible. What are the strengths and weaknesses of the maps you found in Part A and Part B?