5. The Concept of Volume

TIMS Tutor

Figure 1

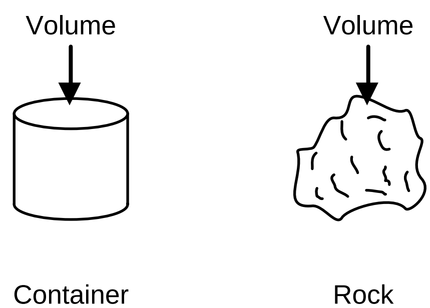
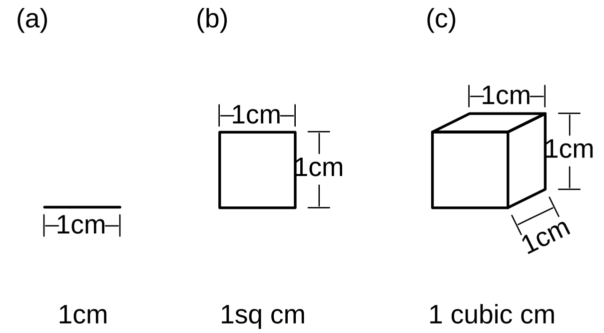


Figure 2



5.1 Defining Volume

Figure 1 shows a container and a rock. The *space* that the container surrounds (and is occupied by air) and the *space* that the rock takes up (and is occupied by elements such as oxygen, silicon, and aluminum) are both called volume. The concept of volume is tricky. Two objects (like our container and rock) might occupy the same volume but might contain totally different amounts of matter. Children often confuse the amount of matter, which we call mass, with the space occupied, which we now know is volume. Thus children tell us that a "heavy" object has more volume than a "light " object even though the latter may actually occupy more space. Indeed, volume is so oversimplified in the elementary schools that most 8th graders we asked thought of volume as length × width × height no matter what the shape of the object. Even worse, some told us that volume was length squared. And so it goes.

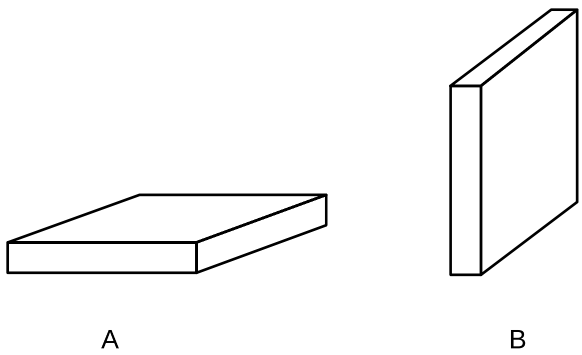
5.2 Units

Since our unit length is the centimeter (Figure 2a), and our unit of area is the extent of the plane surface that is bounded by a square 1cm on a side (Figure 2b), then it is not unreasonable that we take as our unit volume the *space occupied within a cube that is 1cm on a side* (Figure 2c). The volume occupied by such a cube is defined as 1 cubic centimeter whether that volume is occupied by a solid object (Figure 3a) or by empty space (Figure 3b).

Unlike area, it is very hard to divide an object up and count cubic cm. We can't trace the volume the way we trace areas on square cm paper. In theory we could slice it up into 1 cu cm pieces, but this process will destroy the object. Therefore, learning to understand and measure volume can be more difficult than understanding and measuring area.

Figure 3

Figure 4



5.3 Problem of Dimensionality

Another great difficulty in understanding volume is that the concept deals with three dimensions. As Piaget pointed out, it is much easier, and therefore usual, for a child to focus on one dimension. They will decide that a tall object has lots of volume because they only focus on the height and fail to take into account the other two dimensions to make a proper estimate of V. In Figure 4, the two objects have the same volume, but because A is flat and B is upright, children will tell you that B has the greater volume. We shall discuss the whole business of confusing volume for height and an experiment for helping the children come to terms with this problem in the Teacher Lab Discussion of *Marshmallows vs. Containers*, an experiment for 2nd to 3rd graders.

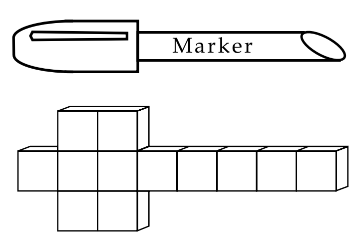
Volume is an extremely important scientific variable. The way it is related to area and to mass and the manner in which it may change with time are all intrinsic to every area of scientific investigation. It is well worth our time to do a good job on volume.

5.4 Measuring Volume—Early Activities

One way to begin to deal with volume, say at the 2nd or early 3rd grade, is by having the children make figures out of a set of Cube-O-Grams. Each Cube-O-Gram is a plastic cube, one cubic cm in volume, which can be joined to other cubes to make a variety of shapes. For example, you can give each child 10 cubes and ask him or her to make a figure whose volume is 10 cubic cm. You will get a variety of shapes, all of which have the same volume. This will begin to impress upon the children the idea that volume is independent of shape, that many different shapes can have the same volume. Another Cube-O-Gram activity to build the children's understanding at this level involves shapes of different volumes. Give each child a few (3–10) Cube-O-Grams. Have each child make a shape with his or her Cube-O-Grams. Then have the children sort themselves into groups according to the volume of their shapes. Any child who tries to join the wrong group will be "driven out" quickly. You can bring in some simple solid shapes, like a piece of chalk, a match box, a pile of washers, etc., and have the children make figures out of cubic cm that approximate the volume (size and shape) of these objects. In this way they can estimate the volume of the original object by keeping track of the number of cubic cm they used. An example is given in Figure 5 of a marking pen and cubic cms linked together to make a shape of approximately the same volume.

The next step would be for you to make several Cube-O-Gram models and have the children count the number of cubic cm in each one. Pictures of 3

Figure 5

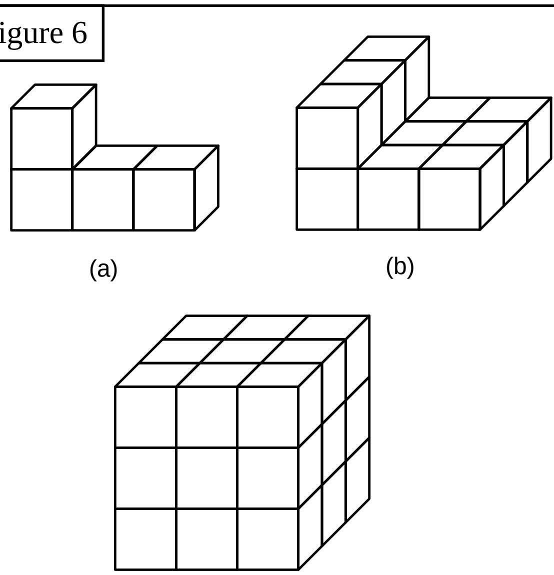


typical models, each a bit more complex than the previous, are given in Figure 6a, b, and c. There are two potential problems here. The children may confuse surface area and volume and count the faces of the cubes calling each face a cubic cm. A subtle problem, exemplified by Figure 6c, is when there are one or more "hidden" cubic centimeters buried inside the figure.

A very basic problem is learning to count the cubic centimeters properly. It is relatively easy to pick up 12 cubic centimeters, make a figure, and say the volume is 12 cu cm. It is quite another to hold a Cube-O-Gram figure like that in Figure 6b, turn it over, and keep track of all the cubic cm. We want the children to learn to be systematic, look for patterns, and use simple basic addition and multiplication to count the cubic cm. For example, Figure 6a is easy since it is only 4 cc. Figure 6b is made up of three pieces, each of which is exactly like Figure 6a. Your students can then use addition to see that the volume is 4 cc + 4 cc + 4 cc, or use multiplication, 3×4 cc. Likewise in Figure 6c, it would be difficult to find and keep track of each cubic cm in Figure 6c. A systematic approach allows the children to solve the problem easily. There are 9 cc in the top layer and there are three layers; therefore the volume is

$$9 \text{ cc} + 9 \text{ cc} + 9 \text{ cc} = 27 \text{ cc}.$$

Figure 6



(a) (b) (c)

To help the children master the skills of counting cubic cm and distinguishing a cubic cm from a square cm, we have two experiments, *Surface Area vs. Shape* and *Surface Area vs. Volume or Why Is the Fly Dry?* Both experiments have interesting physical and biological applications, as discussed in the Teacher Lab Discussions for those experiments.

After the children have mastered finding the number of cubic cm in a three dimensional object, they should learn to count the number of cubic cm in a figure drawn on a piece of paper, like those in Figure 6. This is harder than counting cubic cm in a freestanding object because they cannot pick up the object and look at it from all sides. They must try to do this mentally. Again, many of the students will count only the faces that they see or the cubes that they see and miss the others. For example, in our TIMS test many children said that a figure like that in Figure 6a had a volume of 9 cc, not 4 cc, because they counted the visible faces instead of the cubes. Most math books will have drawings of volumes like those shown in Figure 6. If not, draw up your own based on Cube-O-Gram figures you have made.

Finally, and hardest of all, you can ask the children to try to draw a figure with a given number of cubic cm. It is very hard for anyone to draw a cube, and a figure with several cubic cm is harder still. Nevertheless, it is worth a try since doing so will help them improve their spatial perception and will force them to "think" in three dimensions.

Let us review the 4 steps we have just described:

(1) Make figures out of a given number of Cube-O-Grams.

(2) Count cubic cm in a figure.

(3) Count cubic cm in the drawing of a figure.

(4) Draw a figure with a given number of cubic cm.

The above is an introduction for 2nd and 3rd graders. The most important tool, however, for determining volume will be the graduated cylinder which we will discuss in the section after next. But first we want to talk about how older children can calculate volume.

5.5 Calculating Volume—An Upper Grade Exercise

If the object is a regular array of cubic cm, like the block shown in Figure 7, then one can count the cubic cm by using multiplication. The number of cubic cm in the top layer is just the product of 3×6 since there are 3 rows of 6 cubic cm. Thus in each layer there are 18 cubic cm. Because there are 5 layers the total number of cubic cm is $18 \times 5 = 90$ cc. As often written in math books, this type of counting is expressed as

$$V\,(cc) = l \times w \times h.$$

But you should interpret this as the number of cubic cm in the top layer (given by the value of $l \times w$) times the number of layers (given by the value of h).

If the top layer is not rectangular but the figure is a right solid, then we can still find the volume by the above technique, although the formula *length* × *width* × *height* will no longer work. Since each horizontal slice has the same shape (see Figure 8), all we have to do is find the number of cubic cm in the top layer and multiply this by the number of layers. To find the number of cubic cm in the top layer, we have to find the number of square cm in the top surface, since each square cm of the surface is attached to a cubic cm in the top layer. Thus, if by counting square cm we find that there are 22 sq cm on the top surface, then there must be 22 cubic cm in the top layer and in each subsequent layer. The total volume, then, is

$$\frac{22 \text{ cubic cm}}{1 \text{ layer}} \times 5 \text{ layers} = 110 \text{ cubic cm}$$

for the object shown in Figure 8. As a general formula we have

$$V(cc) = A \times h,$$

where the area A of the top also tells us the number of cc in the top layer, and the value of h also tells us the number of layers. The children should not just memorize each formula. They should understand what is behind the formula.

When using a formula like $l \times w \times h$ the old bug-a-boo of units reappears. Since l, w, and h are all measured in cm, it is tempting to say that the units of volume are centimeters cubed written as $(\text{cm})^3$. And indeed this is what is done again and again.

Figure 7

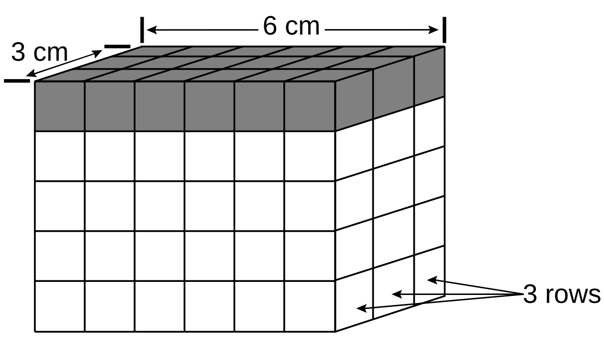
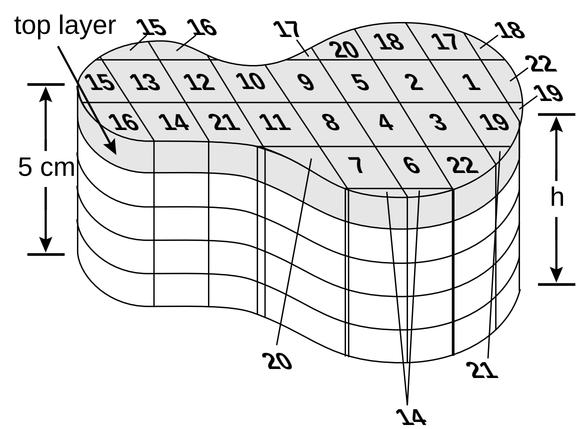


Figure 8



Yes, it's technically correct to write $(cm)^3$ but again it can be misleading for children just as centimeter squared can be misleading for area. If we say a volume is 7 centimeters cubed, is that $7^3 = 343$ cubic centimeters? One avoids all confusion by writing what one means: that the volume is 7 cubic centimeters, period.

Clearly the ability to calculate volume is rather limited to a few special shapes. An odd-shaped container or an odd-shaped object will require a different approach.

5.6 A Volume Measurer: The Graduated Cylinder

We have seen that a ruler which is calibrated in cm can be used to measure length. There is no comparably simple device for measuring area, but there is one for measuring volume. It is the graduated cylinder calibrated in cubic cm. Two are shown in Figure 9. They can be made of glass or pyrex, both of which are breakable, or of plastic which is not. The most suitable for our use would be calibrated in 1 cc, 5 cc, or 10 cc divisions (cc is short for cubic cm) and have a capacity of 100 cc to 150 cc. When filled with a liquid (usually water) or a densely packed substance like sand or salt, one can read the volume of the material off the side of the graduated cylinder. One cannot use this device to measure directly the volume of a number of marbles since the marbles piled in the cylinder will leave an unknown volume of air spaces between them. This is illustrated in Figure 10. As we shall see in Section 5.7, we can find the volume of solid objects like marbles by the method of displacement. For this section we shall stick to filling the cylinder with water.

Figure 9

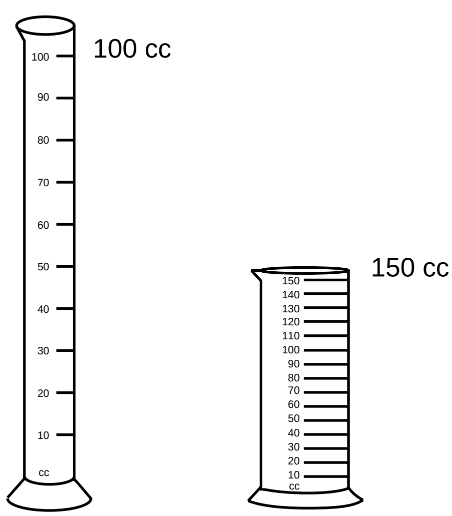
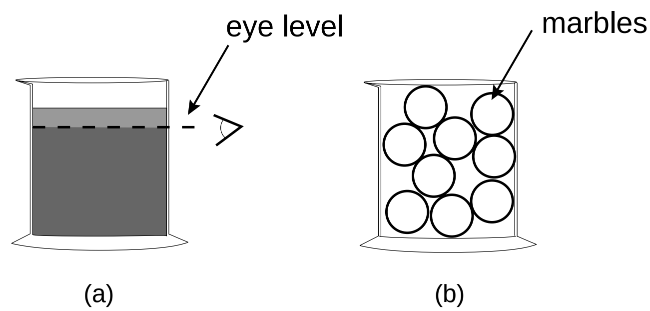


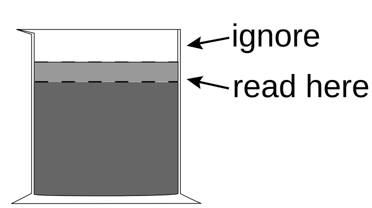
Figure 10



To read the volume of water one must bend down so that your eyes are level with the liquid, as shown in Figure 10a. Since water is pulled up at the sides of the cylinder into a curved surface called a meniscus, one must measure the water level at the center of the cylinder. This is done by using the lower of the two lines that one sees (Figure 11) when looking at the water from the side. The top line is due to the pulled up water and should be ignored. (This phenomenon tends to occur more so when using glass rather than plastic graduated cylinders.)

One must be careful of what one puts into a graduated cylinder. A tall, narrow one will generally have 1 cc divisions and thus can be used

Figure 11



for accurate measurements. The trouble is, because they are narrow only small objects can be used.

Many a time a student has misjudged the size of an object and found it stuck in the narrow cylinder never to come out again. Chalk off one cylinder. A good general size that we like is one about 4 cm in diameter with a 150 cc capacity and made of plastic. The divisions are usually 5 or 10 cc. We shall discuss in the next section how to use a "big" graduated cylinder to measure the volume of small objects.

Most graduated cylinders that you purchase will have several different scales along the sides. This is due to the diversity of units for liquid measurements. There are fluid ounces and milliliters as well as cc. We will not use fluid oz since they are not part of the metric system. Milliliters, on the other hand, present a problem. They are really not a proper set of units for experimental work. They are more colloquial than scientific and should always be converted to cubic cm. The conversion is very simple:

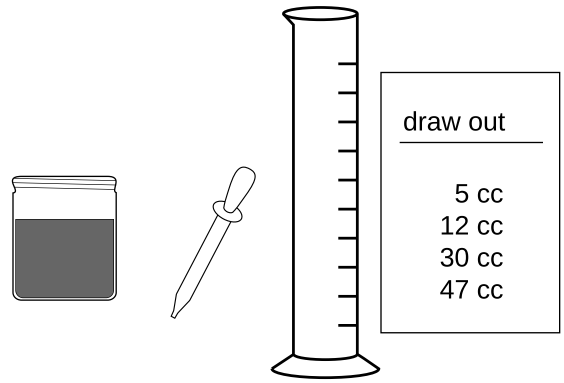
1 milliliter (1 ml) = 1 cubic cm.

If it were up to us, we would take all graduated cylinders and scratch out or paste over milliliters and replace them with cubic cm.

One of the first exercises the children can do with a graduated cylinder is simply to pour a given amount of water into the graduated cylinder (see *Measuring Volume).* An eyedropper is handy for getting the volume exactly right, since by pouring, one usually overshoots or undershoots the mark. Whatever the divisions of the cylinder are, pick out some volumes that fall right on a major division and some that fall between divisions, where the children will have to interpolate. An example is shown in Figure 12 for a cylinder with 5 cc divisions.

Once the 2nd or 3rd graders are good at reading the scale, they can use the graduated cylinder to find the capacity or volume of a set of 3 jars

Figure 12



(Figure 13). You should build up a collection of jars of all shapes and volumes, from small baby food jars, through peanut butter containers, to large jars of coffee. Exotic shapes are nice. One can either fill the jar to the brim and then keep pouring the water into the graduated cylinder, or one can keep filling the graduated cylinder and pour the water into the jar until it is filled. Either way the children have to keep track of the number of times the graduated cylinder is filled and the total volume of water accumulated this way. This is one of our TIMS experiments called *Fill 'er Up* and is discussed in detail in the accompanying Teacher Lab Discussion.

Sometime in the 3rd grade the children should take one of the jars and learn to calibrate it, that is, turn it into a graduated cylinder. To make life simple, let's calibrate a wide-mouthed jar into

Figure 13

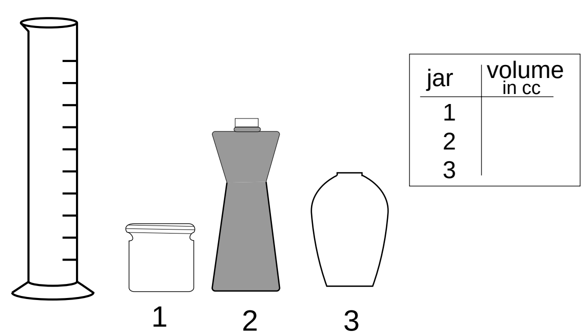
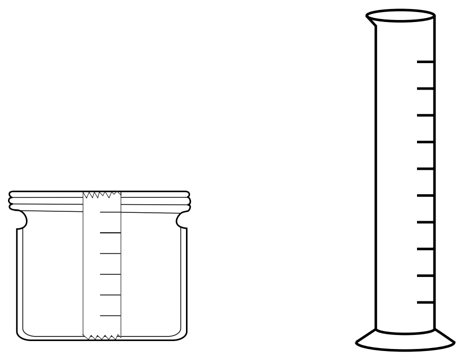


Figure 14



10 cc divisions. First you place a piece of tape down the length of the jar, as shown in Figure 14. Then take 10 cc from the graduated cylinder, pour it into the jar, and mark the lower line of the water level on the tape. This mark is now 10 cc. Then pour the 10 cc out and pour 20 cc in and mark that point. Pour the 20 cc out and pour 30 cc in and mark that point, etc. One could calibrate the jar by pouring 10 cc in, leave it, then pour another 10 cc for a total of 20 cc, then another 10 cc for a total of 30 cc, etc. The trouble with the latter method is greater error. A 1 cc uncertainty in the volume (a not uncommon error) is a 10% error at 10 cc, a 5% error at 20 cc, and only a 2.5% error at 40 cc. Thus, your scale will be more accurate if you pour in large amounts rather than a sequence of small amounts. If you want to extend the calibration of the jar beyond the capacity of the graduated cylinder, then you have to pour in a full graduated cylinder, leave that water, and add to it in the manner described above. The children can then use their calibrated jar to determine the volume of objects that are either too wide to fit into the narrow graduated cylinder or have too large a volume.

5.7 Volume Measurement by Displacement

How would you measure the volume of a rock? By the time the children are through with the 3rd or 4th grade, they should know the answer to that question. The technique is illustrated in Figure 15.

First you fill the graduated cylinder with a convenient amount of water, for example, 40 cc and not 43 cc. Then you place the rock in the graduated cylinder without losing any water (there must be enough water initially to cover the object). You then read the new volume V. Since the volume of water V_w stays constant, the volume V is due to the water plus the rock. The rock displaces, or pushes aside, its volume in water and the water level rises. Thus we have,

$$V_{rock} = V - V_w.$$

For example in Figure 15 the volume of the rock is 22 cc. The subtraction is easier if you start with a multiple of ten for the volume of water. The technique works for any solid object no matter what its shape. To start with, then, the children should be asked to find the volume of a wide variety of objects, some spheres, cubes, rocks, coins, washers, etc., which they do in *Measuring Volume.*

But what if the object is too big to fit into the graduated cylinder? Then you have no choice, you must find a larger one. If a bigger graduated cylinder isn't handy, then you can use your graduated cylinder to calibrate a large jar, and then away you go. One word of warning. When you place a large object in a graduated cylinder or jar, a considerable amount of water may splash out, even though you are very careful. You can get around this by placing the object in the graduated cylinder first and then pouring in a known amount of water V_w, and then read $V = V_{rock} + V_w$ off the

Figure 15

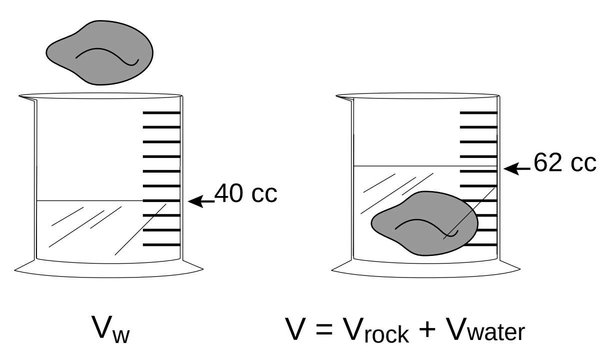
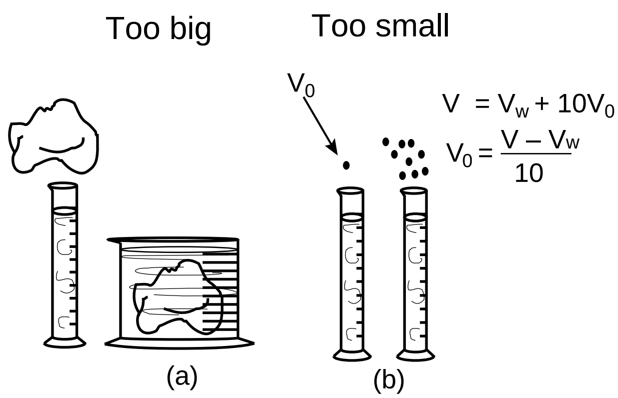


Figure 16



scale.

Sometimes you have the opposite problem: the object is so small you can't see the water level rise. Of course, you can always use a smaller graduated cylinder, one with 1 cc divisions instead of 5 or 10 cc divisions. The trouble is that sometimes even 1 cc per division is too large. The only way out is to measure the volume of several of the small objects at once (Figure 16b). If you have them, 10 is a good number to use. If V_w is the volume of the water initially, then the final volume in the graduated cylinder is $V = V_w + 10V_o$, where V_o is the volume of one of the small objects. The volume of the 10 identical objects is $10V_o = V - V_w$.

Say

$$V - V_w = 3 \text{ cc}.$$

Then we have

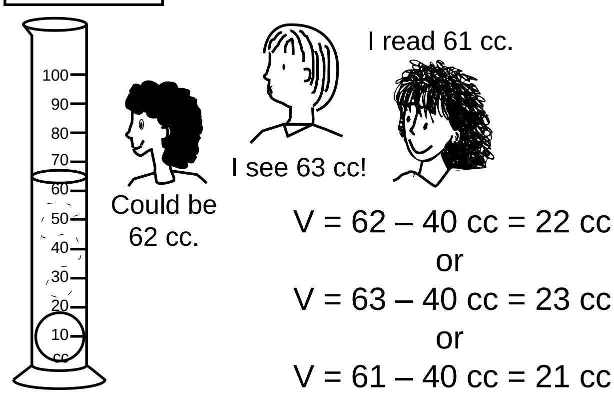
$$V_{obj} = \frac{(V - V_w)}{10} = \frac{3 \text{ cc}}{10} = 0.33 \text{ cc}$$

In this fashion the children can find the volume of small washers, paper clips, pins, etc.

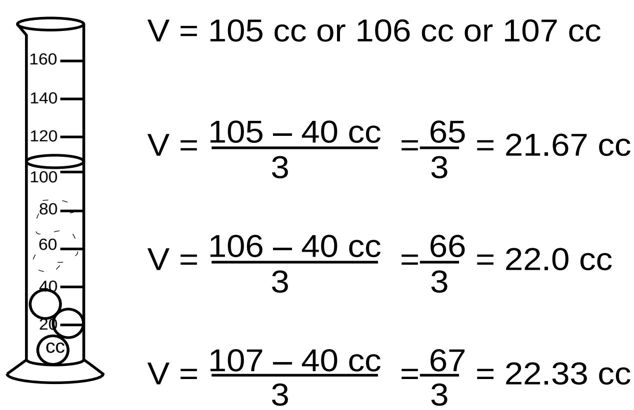
5.8 Obtaining Accurate Results

A few words about accuracy are appropriate here. Most of the time you have to interpolate between divisions in order to read the volume. This usually leads to a reading error of about 20% of the value between the scale marks. Thus, if the graduated cylinder has 10 cc divisions, you might expect an error (ie., children will get readings that differ) by up to 2 cc. This is illustrated in Figure 17a. If the object has raised the water level several divisions, then this 20% per division uncertainty is not a serious problem. For example, our rock changed the water level from 40 cc to 62 cc. For a 5 cc per division graduated cylinder this can lead to a reading error of 2 cc. That is, three children who read the same graduated cylinder might read 62 cc, but possibly 61 cc or 63 cc (if they are careful; maybe even more if they are not). But the volume of the rock is 62 cc – 40 cc = 22 cc for one child, but 23 cc for another, and 21 cc for the third.

Figure 17



(a)



(b)

Figure 18

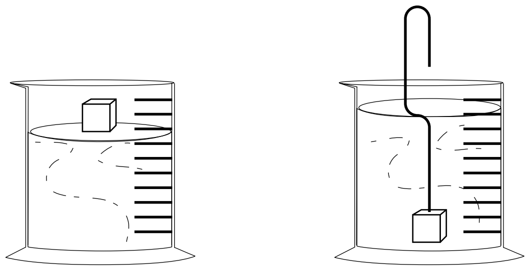
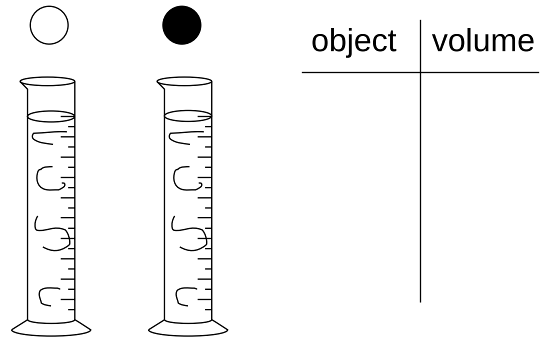


Figure 19



Thus we have a spread in reading of about 2 parts in 22 or 10%. A bigger rock, with a volume of, say, 46 cc would still have the same reading error of 2 cc, but its volume error would only be 2 parts in 46 or about 5%.

A good rule of thumb is that your reading will not be accurate unless the object raises the water level by more than one division. There are two ways to achieve this. One, use a narrow graduated cylinder, or two, use several identical objects if they are available. In the case of the latter, you find the volume of one object by dividing by the number of identical objects, as we did above for very small objects. The reading error for the several objects is still only 2 cc, so the volume error for one object divides by the number of objects. Thus, in general, three objects would still produce a reading error of 2 cc but a volume error of only 2 cc/3 = 0.67 cc. This is illustrated in Figure 17b where three similar rocks give a volume spread of only 0.67 cc in 22 cc or 3%, compared to 10% for one rock. Of course, you are not likely to find three identical rocks. However, some of our experiments do use identical objects. We shall remind you of this problem when we discuss each experiment.

If the object floats, you have to push it under in order to measure its volume. Here, what is important is how you push it under. If you use your finger, then what you measure is the volume of the object plus the volume of your submerged finger. Since the volume of your finger may be comparable to the volume of the object, this is clearly not a good idea. What you need is a pusher whose volume is much less than the volume of the object. A straightened paper clip or a pin will do (Figure 18).

5.9 Two Misconceptions

Early on, the children should come to grips with two important ideas concerning volume:

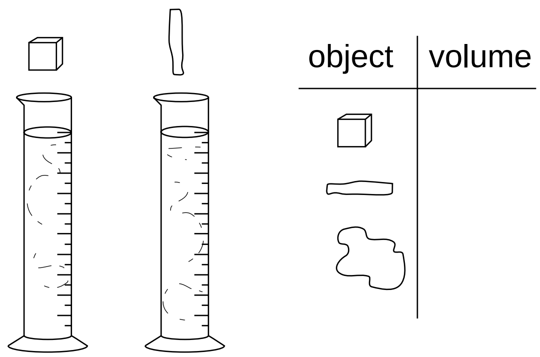
(1) The volume of an object is independent of the material it is made of;

(2) The volume of an object is independent of its shape.

As to the first point, TIMS has several spheres of the same volume but made of a wide variety of materials (steel, lucite, glass, even wood). The children should discover that their volumes are the same. As illustrated in Figure 19, they can fill two identical cylinders to the same level, carefully place both objects in, and see that the volumes are the same. Initially, many children will say that the heavier object has more volume and that the water will go up higher in its graduated cylinder. They are confusing mass and volume. We explore this concept in the experiment *Volume vs. Material.*

If you take a clay cube and mash it into a thin disk, many children will say that the disk has less

Figure 20



volume than the cube. Here they confuse one dimension, thinness, with volume: they mistakenly assume all thin objects have small volume. Again, using the graduated cylinder the child can see that the volumes are the same (Figure 20). They can make all kinds of shapes out of a single piece of clay and determine that they all have the same volume. Indeed, the only way that they can change its volume is to tear off a piece. This exercise is carried out in *Volume vs. Shape*.

Another nice thing about these two experiments is that they tell you if the children are reading the graduated cylinders correctly. Since you know the volumes must be identical, any spread in the volumes by more than 1 or 2 cc means that they are not reading the graduated cylinders accurately.

5.10 Volume Preview

We now preview some TIMS experiments that involve measurement of volume. We start with the experiment *Full of Beans* where different sized beans fill a “standard volume.” That is followed up by the Piagetian exercise *Marshmallows vs. Containers* and these by an introduction to the graduated cylinder in *Measuring Volume*. We hope that between *Full of Beans* and *Measuring Volume* you will have done some of the early activities.

Following these exercises is the early displacement experiment that we discussed in the last section, *Volume vs. Shape*, as well as the early exercise involving filling containers, *Fill ’er Up!*

In contrast to many of the length and area experiments, the experiment *Volume vs. Number* has a graph that gives a straight line not through (0,0). In the experiment the children find how much the volume increases when 2, 4, and 8 marbles are placed in the cylinder with an initial volume of water.

Candle Burning I is an exciting experiment that relates the volume of a jar to the time a candle will burn in the jar. The change in the volume of water in a container over several days is studied in *Evaporation I.* This experiment has an interesting graph: the best-fit curve is a straight line not through (0,0) with negative slope.

Some crucial considerations in biology are studied in *Surface Area vs. Volume or Why Is the Fly Dry?* And in *Volume vs. Diameter* the children tackle a nonlinear graph that is not too dificult to straighten out.

As you can see, many volume experiments have graphs that are a bit different from the straight lines through (0,0) that are so familiar. This makes studying volume doubly worthwhile: the children learn about a fundamental variable and extend their analytic techniques at the same time.

All of these experiments involve volume and a previously learned variable. Later we shall study the concept of mass and see how volume and mass are related. We shall further see the role each plays in whether objects sink or float and what determines the buoyant force on a submerged object. But for now, let’s learn what there is to know about the concept of volume.